FINITE ELEMENT ANALYSIS FOR BENDING OF GEOMETRICALLY AS WELL AS MATERIALLY NON-LINEAR LAMINATED COMPOSITE PLATES USING HSDT

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by

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CERTIFICATE

It is certified that the work contained in this thesis entitled "**Finite Element Analysis for Bending of Geometrically as well as Materially Non-linear Laminated Composite Plates Using HSDT**", by **Arunesh Dixit**, has been carried out under my supervision and that this work has not been submitted elsewhere for any degree.

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Dedicated to

Parents, Sisters

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Abstract

A higher order theory which satisfies zero transverse shear stress conditions on the bounding-planes of a generally laminated fibre-reinforced composite plate subjected to transverse load is used to incorporate with the Von Karman's geometric nonlinearity. A C^0 continuous displacement finite element formulation with seven degrees of freedom, is presented and the coupled membrane-flexure behavior is investigated. The 9-noded Lagrangian elements are used for mesh elements of the plate.The results for isotropic, orthotropic and cross-ply plates are found. The material non-linearity is also introduced to make a complete nonlinear study for bending of the laminated plates.The variation in bending behavior is obtained with a/h by using linear HSDT, geometric nonlinear HSDT and completely nonlinear HSDT. Some cases of angle-ply laminates are also presented. The first-ply failure analysis for various lamination-schemes is also done for transverse loads only, and results are compared with those available in various literature.

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Chapter 1

Introduction and Literature Review

1.1 General

In recent years, advanced composite materials are being used increasingly in many engineering and civilian applications, ranging from the fuselage of an aeroplane to the frame of tennis racket. The high stiffness-to-weight ratio, coupled with the flexibility in the selection of lamination scheme that can be tailored to match the design requirements, make a laminated plates attractive structural components for automotive and aerospace vehicles. The increased used of laminated plates in various structures has created considerable interest in their analysis.

The analysis of plate and shell structures is of considerable interest in various areas of structural mechanics. It is, therefore, natural that with the development of the finite element method a large number of different finite elements have been formulated for the analysis of plate and shell problems. In these developments, basically two approaches have been followed: firstly, what maybe called a classical procedure and, secondly, an approach in which displacement/rotation isoparametric elements are employed.

1.2 Literature review

The two-dimensional analyses of laminated composite plates in the past have been based on one of the following two theories:

(1) the classical lamination theory,

(2) shear deformation theories.

In both these theories it is assumed that the laminate is in the state of plane stress,the individual lamina are linearly elastic, and there is perfect bonding between layers.The classical laminate theory, which is an extension of the classical plate theory(CPT) to laminated plates, ignores the transverse stress components and models a laminate as an equivalent single layer.The first complete laminated anisotropic plate theory is attributed to Reissner and Stavsky[1].The classical laminate theory is adequate for many engineering problems.However, laminated plates made of advanced filamentary composite materials, such as graphite-epoxy (whose elastic modulus to shear modulus ratios are very large), are susceptible to thickness effects because their transverse shear modulii are significantly smaller than the effective elastic modulus along the fibre direction.These high ratios of elastic modulus to shear modulus render the classical laminate theory inadequate for the analysis of composite plates.An adequate theory must account for accurate distribution of transverse shear stresses.

To improve the situation, many theories which account for the transverse shear and normal stresses are available in the literature. The Reissner[2]-Mindlin[3]-type theories for isotropic plates was extended to laminated structures[4].This theory is normally called the First Order Shear Deformation Theory or FSDT. These theories are based on an assumed displacement field that is expanded in terms of the thickness coordinate.A generalisation of the FSDT for homogeneous isotropic plates is due to Yang *et al.* [5] and Whitney and Pagano[6].In this theory, the normals to the mid-plane before deformation remain straight but not necessarily normal to the mid-plane after deformation,and consequently, a correction

to transverse stiffnesses is required. Since the transverse shear strains are constant through the thickness, the transverse stresses are also constant. And therefore, these shear deformation theories do not satisfy the condition of zero transverse shear stresses on the top and bottom surfaces of the plate, and require the shear correction to the transverse shear stiffnesses. Introducing the shear correction factors, whose accurate prediction for anisotropic laminates is cumbersome and problem dependent, J.M.Whitney[7], is not always desirous and pleasing.

Thus in order to have a reliable analysis and safe design, the proposal and developments of models using higher order shear deformation theories have been considered. Lo et al.[8] and [9] reviewed the available work in the field and formulated a theory which accounts for the effects of transverse shear deformation, transverse strain and polynomial distribution of the in-plane displacements with respect to the thickness co-ordinate. The theory showed that for problems which involves rapidly fluctuating loads with characteristic length of the order of the thickness of plate, one of those polynomial theories is required to obtain the meaningful results.

Third order theories have been developed by many theoreticians e.g. Reddy[10], Librescu[11], Schmidt[12], Murty[13], Levinson[14], Seide[15], Murthy[16], Bhimaraddi and Stevens[17], Mallikarjuna and Kant[18], Kant and Pandya[19], and Phan and Reddy[20] among many others. Third order theories are simplest of the all the higher order theories, yet the desired degree of accuracy is very good.So the third order theories are very popular when computational complexities are desired to be avoided. In higher order theories, an additional dependent unknown is introduced into the theory with each additional power of the thickness coordinate. In addition,these shear deformation theories do not satisfy the condition of zero transverse shear stresses on the top and bottom surfaces of the plate, and require a shear correction to the transverse shear stiffnesses. The three dimensional theories of laminates, in which each layer is treated as homogeneous anisotropic medium(see Ref.[21])

are intractable as number of layers become large. Thus, a simple two-dimensional theory of plates that accurately describes the global behaviour of laminated plates seems to be a compromise between accuracy and ease of analysis.

The displacement field used by Reddy[22] is similar to that of Levinson[14] and Murthy[[16], but the latter uses the equilibrium equations of the classical plate theory,which are inconsistent with the assumed displacement field.Reddy's modifications consist of a more systematic derivation of displacement field and variationally consistent derivation of the equilibrium equations.The conventional variational formulation of the classical plate theory as well as higher-order theory involves higher order(i.e. second order) derivatives of the transverse displacement.Therefore in finite element modelling of such theories one should impose the continuity of not only the transverse displacement but also its derivatives along the element boundary.In other words, a conforming plate bending element based on displacement formulation of these theories requires continuity of transverse displacement and their derivatives across the inter-element boundaries [23].The construction of such an element is algebraically complicated requiring, for example, a quintic polynomial with 21 degrees of freedom for a six-node triangular element .Computationally the element requires much storage and computer time.

To overcome the stringent continuity requirements placed by the conventional variational formulation of the classical plate theory , several alternative formulations and associated elements have been proposed(see(Putcha and Reddy[24])). These include the hybrid finite elements and mixed finite elements.The hybrid elements are based on variational statements that use independent variation of displacements inside the domain of the element and tractions on the boundary of the element.The mixed elements use stationary variational principles, such as the Reissner variational principle or Hu-Washizu variational principle, to construct independent variations of both displacements and bending moments in a plate.The conventional variational formulation of the FSDT leads to a C^0-element, often referred to as

the Mindlin element.Thus Mindlin plate elements are particularly easy to formulate. Such formulations readily accomodate moderately thick or thin plates of variable thickness, curved boundaries and laminated plates.The initial confidence expressed in these elements was later subject to doubts on two accounts.First, the serendipity elements were found to exhibit an overstiff locking behaviour in very thin plate situations and secondly, the Lagrangian elements with reduced integration rules contained spurious zero energy modes not associated with rigid body movement.However, extensive studies (Pugh *et al.*,1978) on this phenomenon led to the conclusion that the nine-noded Lagrangian element with selective integration of the shear terms is the best for such elements.

The present study uses the Higher Order Shear Deformation theory of Kant and Pandya[25] But this study differs from the previous theories in two ways:

(1) it incorporates Von Karmans' non-linearities terms,and

(2) it considers non-linear constitutive laws too.

The geometric non-linearity is incorporated into the conventional HSDT formulation proposed by Kant *et al.*[25].This makes the finite element formulation nonlinear.The boundary conditions are of mixed type and the governing differential equations are non-linear.A nine-noded Lagrangian C^0-continous element with seven degrees of freedom per node is proposed. The solutions to problems with material nonlinearity were attempted by numerous researchers (see Ref.[26]),for shells and plates.The analysis of laminated composite plates with material nonlinearity was done by Reddy *et al.*[27]. In which, the modified Ramberg-Osgood[28] formulation is utilised for the calculation of tangent modulus.In another work, a cubical formula is proposed for the shear stress calculation only, and linear stress-strain relation is used for the normal stress calculations[29].

Petit and Waddoups[30] simulated the nonlinear stress-strain response of a lamina by using an increment method. Hahn and Tsai[31], by using complementary elastic energy density, presented a constitutive equation for in-plane shear stress and strain of a composite

lamina. Sandhu[32], by using cubic spline interpolation functions, represented experimental shear stress-strain data. Jones and Nelson[33] modeled nonlinear mechanical properties as functions of strain energy density. Jones and Morgan[34] showed that Jones-Nelson model converged only up to a specific strain energy value.

In the present work,we utilized both of these approaches [28] and [29], so that, for the normal stresses,the modified Ramberg-Osgood formula is used and for tangential stresses, the cubical stress-strain formula is used upto the limit of practicality of stresses. Beyond the limit of practicality of the stresses, the structure is seldom encountered with larger stresses.

Chapter 2

Model Problem

2.1 Kinematics:

The development of the present HSDT begins with the assumption of the displacement field in the following form:

$$u(x, y, z) = u_0(x, y) + z * \theta_x(x, y) + z^2 * u_0^*(x, y) + z^3 * \theta_x^*(x, y) \tag{2.1}$$

$$v(x, y, z) = v_0(x, y) + z * \theta_y(x, y) + z^2 * v_0^*(x, y) + z^3 * \theta_y^*(x, y) \tag{2.2}$$

$$w(x, y, z) = w_0(x, y) \tag{2.3}$$

where u_0, v_0 and w_0 denote the displacements of a point(x,y) on the mid-plane and $\theta(x)$ and $\theta(y)$ are the rotations of the normals to the mid-plane about y- and x-axes respectively. The parameters u_0^*, v_0^*, θ_x^* and θ_y^* are the corresponding higher order terms in the Taylor's series expansion and are also defined at the mid-plane. The condition that the transverse shear stresses vanish on the top and bottom faces of the laminate is equivalent to the requirement that the corresponding strains be zero on these faces.

The transverse shear strains are given by:

$$\gamma_{yz} = \frac{\partial(v)}{\partial(z)} + \frac{\partial(w)}{\partial(y)} = \theta_y + 2 * z * v_0^* + 3 * z^2 * \theta_y^* + \frac{\partial(w_0)}{\partial(y)} \tag{2.4}$$

$$\gamma_{xz} = \frac{\partial(u)}{\partial(z)} + \frac{\partial(w)}{\partial(x)} = \theta_x + 2 * z * u_0^* + 3 * z^2 * \theta_x^* + \frac{\partial(w_0)}{\partial(x)} \tag{2.5}$$

Now, equating,

$$\gamma_{yz} = \gamma_{xz} = 0 \qquad at(x, y, \pm(h/2))$$

we get,

$$v_0^* = u_0^* = 0 \tag{2.6}$$

and

$$\theta_y^* = \frac{-4}{3h^2} * (\theta_y + \frac{\partial(w_0)}{\partial(y)}) \tag{2.7}$$

$$\theta_x^* = \frac{-4}{3h^2} * (\theta_x + \frac{\partial(w_0)}{\partial(x)}) \tag{2.8}$$

using these relationships, we get the new displacement field as:

$$u = u_0 + z * \theta_x + z^3 * \theta_x^* \tag{2.9}$$

$$v = v_0 + z * \theta_y + z^3 * \theta_y^* \tag{2.10}$$

$$w = w_0 \tag{2.11}$$

Now, we shall proceed with displacement field given by equations(9,10,11), and later we shall introduce conditions (7,8) into the shear-rigidity matrix,D_s. The strains associated with the displacement field are:

$$\epsilon_{xx} = \epsilon_{x_0} + z * \kappa_{xx} + z^3 * \kappa^*_{xx} \tag{2.12}$$

$$\epsilon_{yy} = \epsilon_{y_0} + z * \kappa_{yy} + z^3 * \kappa^*_{yy} \tag{2.13}$$

$$\gamma_{xy} = \gamma_{xy_0} + z * \kappa_{xy} + z^3 * \kappa^*_{xy} \tag{2.14}$$

$$\gamma_{yz} = \phi_{yy} + z^2 * \phi^*_{yy} \tag{2.15}$$

$$\gamma_{xz} = \phi_{xx} + z^2 * \phi^*_{xx} \tag{2.16}$$

Here, we shall introduce the Von Karmans' non-linearity terms to formulate the geometric non-linearity problem.Therefore,

$$\epsilon_{x_0} = \frac{\partial u_0}{\partial x} + \frac{1}{2} * (\frac{\partial w}{\partial x})^2 = \epsilon^L_{xx} + \epsilon^{NL}_{xx} \tag{2.17}$$

$$\epsilon_{y_0} = \frac{\partial v_0}{\partial y} + \frac{1}{2} * (\frac{\partial w}{\partial y})^2 = \epsilon^L_{yy} + \epsilon^{NL}_{yy} \tag{2.18}$$

$$\gamma_{xy_0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x} * \frac{\partial w}{\partial y} = \gamma^L_{xy} + \gamma^{NL}_{xy} \tag{2.19}$$

$$\kappa_{xx} = \frac{\partial \theta_x}{\partial x}; \kappa_{yy} = \frac{\partial \theta_y}{\partial y}; \kappa_{xy} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \tag{2.20}$$

$$\kappa^*_{xx} = \frac{\partial \theta^*_x}{\partial x}; \kappa^*_{yy} = \frac{\partial \theta^*_y}{\partial y}; \kappa^*_{xy} = \frac{\partial \theta^*_x}{\partial y} + \frac{\partial \theta^*_y}{\partial x} \tag{2.21}$$

$$\phi_{xx} = \theta_x + \frac{\partial w_0}{\partial x}; \phi_{yy} = \theta_y + \frac{\partial w_0}{\partial y}; \phi^*_{xx} = 3.0 * \theta^*_x; \phi^*_{yy} = 3.0 * \theta^*_y \tag{2.22}$$

Now, the strain vector can be written as,

$$\{\epsilon\} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{array} \right\} = \{\epsilon^L\} + \{\epsilon^{NL}\} + z * \{\kappa\} + z^2 * \{\phi^*\} + z^3 * \{\kappa^*\} + \{\phi\} \tag{2.23}$$

where,

$$\{\epsilon^L\} = \left\{ \epsilon_{x0}, \quad \epsilon_{y0}, \quad \gamma_{xy_0}, \quad 0, \quad 0 \right\}^T \tag{2.24}$$

$$\{\epsilon^{NL}\} = \left\{ 0.5 * (\frac{\partial w}{\partial x})^2, \quad 0.5 * (\frac{\partial w}{\partial y})^2, \quad (\frac{\partial w}{\partial x} * \frac{\partial w}{\partial y}), \quad 0, \quad 0 \right\}^T \tag{2.25}$$

$$\{\kappa\} = \left\{ \kappa_{xx}, \quad \kappa_{yy}, \quad \kappa_{xy}, \quad 0, \quad 0 \right\}^T; \ \{\kappa^*\} = \left\{ \kappa^*_{xx}, \quad \kappa^*_{yy}, \quad \kappa^*_{xy}, \quad 0, \quad 0 \right\}^T \tag{2.26}$$

$$\{\phi\} = \left\{ 0, \quad 0, \quad 0, \quad \phi_{yy}, \quad \phi_{xx} \right\}^T; \ \{\phi^*\} = \left\{ 0, \quad 0, \quad 0, \quad \phi^*_{yy}, \quad \phi^*_{xx} \right\}^T \tag{2.27}$$

2.1.1 Constitutive equations:

The constitutive equation for the L^{th} layer can be written with reference to the lamina co-ordinates (1,2,3) as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^L * \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}^L \tag{2.28}$$

$$\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix}^L * \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \tag{2.29}$$

The following relations hold between these and the engineering elastic constants:

$$C_{11} = \frac{E_1}{1 - \nu_{12} * \nu_{21}} \tag{2.30}$$

$$C_{12} = \frac{\nu_{12} * E_1}{1 - \nu_{12} * \nu_{21}} \tag{2.31}$$

$$C_{22} = \frac{E_2}{1 - \nu_{12} * \nu_{21}} \tag{2.32}$$

$$C_{33} = G_{12} \tag{2.33}$$

$$C_{44} = G_{23} \tag{2.34}$$

$$C_{55} = G_{13} \tag{2.35}$$

Now, in the laminate co-ordinates (x,y,z), the stress and strain relationships for the L^{th} lamina are:

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{array}\right\}^L=\left[\begin{array}{ccc}Q_{11}&Q_{12}&Q_{13}\\ Q_{12}&Q_{22}&Q_{23}\\ Q_{13}&Q_{23}&Q_{33}\end{array}\right]^L*\left\{\begin{array}{c}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{array}\right\}^L \tag{2.36}$$

$$\left\{\begin{array}{c}\tau_{yz}\\ \tau_{xz}\end{array}\right\}^L=\left[\begin{array}{cc}Q_{44}&Q_{45}\\ Q_{45}&Q_{55}\end{array}\right]^L*\left\{\begin{array}{c}\gamma_{yz}\\ \gamma_{xz}\end{array}\right\}^L \tag{2.37}$$

where,

$$\{\sigma\}=\left\{\ \sigma_{xx},\ \ \sigma_{yy},\ \ \tau_{xy},\ \ \tau_{yz},\ \ \tau_{xz}\ \right\}^T \tag{2.38}$$

$$\{\epsilon\}=\left\{\ \epsilon_{xx},\ \ \epsilon_{yy},\ \ \gamma_{xy},\ \ \gamma_{yz},\ \ \gamma_{xz}\ \right\}^T \tag{2.39}$$

are stress and strain vectors with respect to laminate axes and Q_{ij} are the plane stress reduced elastic constants in the laminate axes of the L^{th} lamina.

2.1.2 Material model:

If the material used in the laminate construction is non-linear, i.e. the stress and strain relationships are non-linear,then the co-efficients Q_{ij} vary with the deformation.If it is assumed that the nonlinearity can be represented by uncoupled stress - strain behaviour for each of the strain components, then a suitable curve fitted to the experimentally observed stress - strain behaviour can serve as the material constitutive law. The Ramberg - Osgood relationship is one such procedure.

For each of the normal strain components , the stress - strain relationship can be simulated by the Ramberg - Osgood relation as:

$$\epsilon = \frac{\sigma}{E_i} + \lambda * \left(\frac{\sigma}{E_i}\right)^m \tag{2.40}$$

where ϵ is the strain, σ is the stress,E_i is the initial modulus, and λ and m are the parameters defining the curve as derived at two different values of strains in the stress-strain curve.

A modified stress - strain relationship can be derived from the above equation (2.40) as in ([27]),

$$\sigma = \frac{(E_i - Ep) * \epsilon}{\left[1 + \frac{E_i - E_p}{\sigma_p} * \epsilon^m\right]^{1/m}} + \sigma_p \tag{2.41}$$

where E_p is the elasticity modulus at the plastic range and σ_p is the stress at which the tangent to stress-strain curve, at some value of strain in the plastic range, hits the stress-axis. E_p is the angle of this tangent with strain-axis. Tangent modulus can be defined as,

$$E_T = \frac{\partial \sigma}{\partial \epsilon} = \frac{(E_i - Ep)}{\left[1 + \frac{E_i - E_p}{\sigma_p} * \epsilon^m\right]^{(m+1)/m}} + \sigma_p \tag{2.42}$$

For the tangential strain components, the stress - strain relationship can be written as, for the smaller values of strains i.e. upto around 2.5 percent,:

$$\tau_{xy} = G_{xy} * \gamma_{xy} - \lambda_6 * G_{xy} * \gamma_{xy}^3 \tag{2.43}$$

$$\tau_{yz} = G_{yz} * \gamma_{yz} - \lambda_4 * G_{yz} * \gamma_{yz}^3 \tag{2.44}$$

$$\tau_{xz} = G_{xz} * \gamma_{xz} - \lambda_5 * G_{xz} * \gamma_{xz}^3 \tag{2.45}$$

in which λ_6, λ_5 and λ_4 denote the degree of nonlinearity with different shear strains and can be found from experimental curves [31] to suit the purpose of applicability.

The tangent moduli for these shear curves are:

$$E_T' = \frac{\partial \tau'}{\partial \gamma'} = G' - 3 * \lambda' * G' * \gamma'^2 \tag{2.46}$$

where,

$$\{E_T'\} = \left\{ \begin{array}{ccc} E_{T_{xy}}, & E_{T_{yz}}, & E_{T_{xz}} \end{array} \right\}^T \tag{2.47}$$

$$\{\tau'\} = \left\{ \begin{array}{ccc} \tau_{xy}, & \tau_{yz}, & \tau_{xz} \end{array} \right\}^T \tag{2.48}$$

$$\{\gamma'\} = \left\{ \begin{array}{ccc} \gamma_{xy}, & \gamma_{yz}, & \gamma_{xz} \end{array} \right\}^T \tag{2.49}$$

$$\{G'\} = \left\{ \begin{array}{ccc} G_{xy}, & G_{yz}, & G_{xz} \end{array} \right\}^T \tag{2.50}$$

$$\{\lambda'\} = \left\{ \begin{array}{ccc} \lambda_6, & \lambda_5, & \lambda_4 \end{array} \right\}^T \tag{2.51}$$

Equations (2.42) and (2.46) are used for each strain level to update the stiffness matrix, which shall be found in finite element formulation.

2.2 Total potential energy and variational formulation:

The first variation in total potential energy Π of the given system should be zero for the equilibrium of the system:

$$\delta\Pi = 0 \tag{2.52}$$

The displacement vector d is defined as:

$$\{d\}=\left\{\ u_0,\ \ v_0,\ \ w_0,\ \ \theta_x,\ \ \theta_y\ \ \theta_x^*,\ \ \theta_Y^*\ \right\}^T \tag{2.53}$$

The functional given by total potential equation is minimized while carrying out the explicit integration through the thickness.This leads to the following variational formulation:

$$\int_V \delta\epsilon^T \cdot \sigma\, dV - \int_A \delta d^T \cdot P\, dA = 0 \tag{2.54}$$

The internal virtual work is:

$$\delta U_i = \int_V \delta\epsilon^T \cdot \sigma\, dV \tag{2.55}$$

or,

$$\begin{aligned}\delta U_i \ \ = \ \ & \int_A [\delta(\epsilon_{xx}^L + \epsilon_{xx}^{NL}) \cdot N_x + \delta(\epsilon_{yy}^L + \epsilon_{yy}^{NL}) \cdot N_y + \delta(\gamma_{xy}^L + \gamma_{xy}^{NL}) \cdot N_{xy} + \\ & \delta\kappa_{xx} \cdot M_x + \delta\kappa_{yy} \cdot M_y + \delta\kappa_{xy} \cdot M_{xy} + \delta\kappa_{xx}^* \cdot M_x^* + \delta\kappa_{yy}^* \cdot M_y^* + \\ & \delta\kappa_{xy}^* \cdot M_{xy}^* + \delta\phi_{xx} \cdot Q_x + \delta\phi_{yy} \cdot Q_y + \delta\phi_{xx}^* \cdot Q_x^* + \delta\phi_{yy}^* \cdot Q_y^*]\, dA\end{aligned} \tag{2.56}$$

where the following 13 stress - resultants for the n-layered laminate are given as:

$$\left\{\begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array}\right\} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array}\right\}_L \cdot dz \tag{2.57}$$

$$\left\{\begin{array}{ccc} M_x & : & M_x^* \\ M_y & : & M_y^* \\ M_{xy} & : & M_{xy}^* \end{array}\right\} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array}\right\}_L \left[\ z\ :\ z^3\ \right] \cdot dz \tag{2.58}$$

$$\left\{\begin{array}{ccc} Q_x & : & Q_x^* \\ Q_y & : & Q_y^* \end{array}\right\} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \left\{\begin{array}{c} \tau_{xz} \\ \tau_{yz} \end{array}\right\}_L \left[\ 1\ :\ z^2\ \right] \cdot dz \tag{2.59}$$

After the integration of these equations and using stress - strain relationships, we have in matrix notation:

$$\left\{ \begin{array}{c} \underline{N} \\ M \\ \\ \underline{M^*} \\ Q \\ Q^* \end{array} \right\} = \begin{bmatrix} [A] & [B] & [0] \\ \\ \left[B^T\right] & [D_b] & [0] \\ \\ [0] & [0] & [D_s] \end{bmatrix} \left\{ \begin{array}{c} \underline{\epsilon} \\ \kappa \\ \\ \underline{\kappa^*} \\ \phi \\ \phi^* \end{array} \right\} \cdot dz \tag{2.60}$$

in which,

$$N = \left\{ \begin{array}{ccc} N_x, & N_y, & N_{xy} \end{array} \right\}^T \tag{2.61}$$

$$M = \left\{ \begin{array}{ccc} M_x, & M_y, & M_{xy} \end{array} \right\}^T ; M^* = \left\{ \begin{array}{ccc} M_x^*, & M_y^*, & M_{xy}^* \end{array} \right\}^T \tag{2.62}$$

$$Q = \left\{ \begin{array}{cc} Q_x, & Q_y \end{array} \right\}^T ; Q^* = \left\{ \begin{array}{cc} Q_x^*, & Q_y^* \end{array} \right\}^T \tag{2.63}$$

$$\epsilon = \left\{ \begin{array}{ccc} \epsilon_{xx}, & \epsilon_{yy}, & \gamma_{xy} \end{array} \right\}^T ; \kappa = \left\{ \begin{array}{ccc} \kappa_{xx}, & \kappa_{yy}, & \kappa_{xy} \end{array} \right\}^T ; \kappa^* = \left\{ \begin{array}{ccc} \kappa_{xx}^*, & \kappa_{yy}^*, & \kappa_{xy}^* \end{array} \right\}^T \tag{2.64}$$

$$\phi = \left\{ \begin{array}{cc} \phi_{xx}, & \phi_{yy} \end{array} \right\}^T ; \phi^* = \left\{ \begin{array}{cc} \phi_{xx}^*, & \phi_{yy}^* \end{array} \right\}^T \tag{2.65}$$

and,

$$[A]_{3\times3} = \sum_{L=1}^{n} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{13}H_1 \\ \text{SYM.} & Q_{22}H_1 & Q_{23}H_1 \\ & & Q_{33}H_1 \end{bmatrix}^{L^{th\,layer}} \tag{2.66}$$

$$[B]_{3\times6} = \sum_{L=1}^{n} \begin{bmatrix} Q_{11}H_2 & Q_{12}H_2 & Q_{13}H_2 & Q_{11}H_4 & Q_{12}H_4 & Q_{13}H_4 \\ Q_{12}H_2 & Q_{22}H_2 & Q_{23}H_2 & Q_{12}H_4 & Q_{22}H_4 & Q_{23}H_4 \\ Q_{13}H_2 & Q_{23}H_2 & Q_{33}H_2 & Q_{13}H_4 & Q_{23}H_4 & Q_{33}H_4 \end{bmatrix}^{L^{th\,layer}} \tag{2.67}$$

$$[D_b]_{6\times6} = \sum_{L=1}^{n} \begin{bmatrix} Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 \\ Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_5 \\ Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_3 & Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_5 \\ Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 & Q_{11}H_7 & Q_{12}H_7 & Q_{13}H_7 \\ Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_5 & Q_{12}H_7 & Q_{22}H_7 & Q_{23}H_7 \\ Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_5 & Q_{13}H_7 & Q_{23}H_7 & Q_{33}H_7 \end{bmatrix}^{L^{th}\,layer} \tag{2.68}$$

$$[D_s]_{4\times4} = \sum_{L=1}^{n} \begin{bmatrix} Q_{55}H & Q_{45}H & 0 & 0 \\ & Q_{44}H & 0 & 0 \\ \text{Sym.} & & Q_{55}H^* & Q_{45}H^* \\ & & & Q_{44}H^* \end{bmatrix}^{L^{th}\,layer} \tag{2.69}$$

wherein;

$$\begin{cases} H_1 & = & (h_L - h_{L-1}) \\ H_2 & = & \frac{1}{2}(h_L^2 - h_{L-1}^2) \\ H_3 & = & \frac{1}{3}(h_L^3 - h_{L-1}^3) \\ H_4 & = & \frac{1}{4}(h_L^4 - h_{L-1}^4) \\ H_5 & = & \frac{1}{5}(h_L^5 - h_{L-1}^5) \\ H_7 & = & \frac{1}{7}(h_L^7 - h_{L-1}^7) \\ H & = & (H_1 - H_3 \times 4/h^2) \\ H^* & = & (H_5 - H_3 \times h^2/4) \end{cases} \tag{2.70}$$

where h is the overall thickness and h_L is the thickness of L_{th} layer.

The shear-rigidity matrix is evolved by incorporating an alternate form of the equations (2.15) and (2.16),viz.,

$$\begin{cases} \phi_{yy} + \frac{h^2}{4}\phi_{yy}^* & = & 0 \\ \\ \phi_{xx} + \frac{h^2}{4}\phi_{xx}^* & = & 0 \end{cases} \tag{2.71}$$

and using these relations (2.71), the resulting theory becomes consistent in the sense that it satisfies zero transverse shear stress conditions on the bounding-planes of the plate.

2.3 Governing equations and boundary conditions:

From equations (2.24)-(2.27) and (2.56), we have

$$\begin{aligned}
\delta U_i = & \int_A [\{(\delta u_{0,x} \cdot N_x + \delta u_{0,y} \cdot N_{xy}) + (\delta v_{0,y} \cdot N_y + \delta v_{0,x} \cdot N_{xy})\} \\
& + \{(N_x \cdot w_{0,x} \cdot \delta w_{0,x} + N_y \cdot w_{0,y} \cdot \delta w_{0,y}) \\
& + (N_{xy} \cdot w_{0,x} \cdot \delta w_{0,y} + N_{xy} \cdot w_{0,y} \cdot \delta w_{0,x} + Q_x \cdot \delta w_{0,x} \\
& + Q_y \cdot \delta w_{0,y})\} + \{(M_x \cdot \delta \theta_{x_x} + M_{xy} \cdot \delta \theta_{x_y} + Q_x \cdot \delta \theta_x) \\
& + (M_y \cdot \delta \theta_{y,y} + M_{xy} \cdot \delta \theta_{y,x} + Q_y \cdot \delta \theta_y)\} \\
& + \{(M_x^* \cdot \delta \theta_{x,x}^* + M_{xy}^* \cdot \delta \theta_{x,y}^* + 3 \cdot Q_x^* \cdot \delta \theta_x^*) \\
& + (M_y^* \cdot \delta \theta_{y,y}^* + M_{xy}^* \cdot \delta \theta_{y,x}^* + 3 \cdot Q_y^* \delta \theta_y^*)\}] \, dA
\end{aligned} \tag{2.72}$$

And , the external virtual work is:

$$\delta W_e = \int_A \delta d^T \cdot P \, dA \tag{2.73}$$

In equation (2.73) , the load vector P consists of two parts, viz.,

(i) transverse load, P_q, and

(ii) in-plane loads, P_x, P_y, P_{xy}

In bending problem , we shall consider only the transverse loads. Therefore , load-vector is:

$$\{P\} = \{\ 0 \ \ 0 \ \ P_q \ \ 0 \ \ 0 \ \ 0 \ \ 0\ \}^T \tag{2.74}$$

and displacement vector is given by equation (2.53). Therefore , from equations (2.53), (2.73) and (2.74), we have

$$\delta W_e = \int_A \delta w_0 \cdot P_q \, dA \tag{2.75}$$

Therefore , for equilibrium

$$\delta U_i - \delta W_e = 0 \tag{2.76}$$

$$\begin{aligned}
\Rightarrow \int_A \{\{(\delta u_{0,x} \cdot N_x + \delta u_{0,y} \cdot N_{xy}) + (\delta v_{0,y} \cdot N_y + \delta v_{0,x} \cdot N_{xy})\} \\
+\{(N_x \cdot w_{0,x} \cdot \delta w_{0,x} + N_y \cdot w_{0,y} \cdot \delta w_{0,y}) \\
+(N_{xy} \cdot w_{0,x} \cdot \delta w_{0,y} + N_{xy} \cdot w_{0,y} \cdot \delta w_{0,x} + Q_x \cdot \delta w_{0,x} \\
+Q_y \cdot \delta w_{0,y})\} + \{(M_x \cdot \delta\theta_{x_x} + M_{xy} \cdot \delta\theta_{x_y} + Q_x \cdot \delta\theta_x) \\
+(M_y \cdot \delta\theta_{y,y} + M_{xy} \cdot \delta\theta_{y,x} + Q_y \cdot \delta\theta_y)\} \\
+\{(M_x^* \cdot \delta\theta_{x,x}^* + M_{xy}^* \cdot \delta\theta_{x,y}^* + 3 \cdot Q_x^* \cdot \delta\theta_x^*) \\
+(M_y^* \cdot \delta\theta_{y,y}^* + M_{xy}^* \cdot \delta\theta_{y,x}^* + 3 \cdot Q_y^* \delta\theta_y^*)\} - (P_q \cdot \delta w_0)\}\, dA = 0
\end{aligned} \tag{2.77}$$

Integrating the expressions in equation (2.77) by parts, and collecting the coefficients of δu_0,δv_0, δw_0,$\delta\theta_x$,$\delta\theta_y$,$\delta\theta_x^*$ and $\delta\theta_y^*$, we obtain the following euilibrium equations in the domain Ω :

$$\left\{\begin{array}{lllll}
\delta u_0 & : & \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} & = & 0 \\
\delta v_0 & : & \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} & = & 0 \\
\delta w_0 & : & N_x \frac{\partial^2 w_0}{\partial x^2} + 2 \cdot N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} + N_y \frac{\partial^2 w_0}{\partial y^2} & & \\
 & & + \frac{\partial w_0}{\partial x}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right) & & \\
 & & + \frac{\partial w_0}{\partial y}\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}\right) & & \\
 & & + \frac{\partial Q_x}{\partial x} \frac{\partial Q_y}{\partial y} - P_q & = & 0 \\
\delta\theta_x & : & \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x & = & 0 \\
\delta\theta_y & : & \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_y & = & 0 \\
\delta\theta_x^* & : & \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3 \cdot Q_x^* & = & 0 \\
\delta\theta_y^* & : & \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3 \cdot Q_y^* & = & 0
\end{array}\right. \tag{2.78}$$

The boundary conditions shall be of the form:

$$\left\{\begin{array}{lll} u_0 & or & (N_x + N_{xy}) \\ v_0 & or & (N_y + N_{xy}) \\ w_0 & or & \left(\begin{array}{c} N_x \cdot \frac{\partial w_0}{\partial x} \\ +N_y \cdot \frac{\partial w_0}{\partial y} \\ +N_{xy}(\frac{\partial w_0}{\partial x} \\ +\frac{\partial w_0}{\partial y}) \\ +Q_x + Q_y \end{array}\right) \\ \theta_x & or & (M_x + M_{xy}) \\ \theta_y & or & (M_y + M_{xy}) \\ \theta_x^* & or & (M_x^* + M_{xy}^*) \\ \theta_y^* & or & (M_y^* + M_{xy}^*) \end{array}\right. \tag{2.79}$$

This completes the derivation of governing equations.In the present investigation we have taken two types of boundary conditions, e.g. (i)simply-supported boundary at all edges and (ii)clamped boundary at all edges.The boundary conditions as in figure 2.1 are shown only for the quarter plate due to symmetry of the plate.

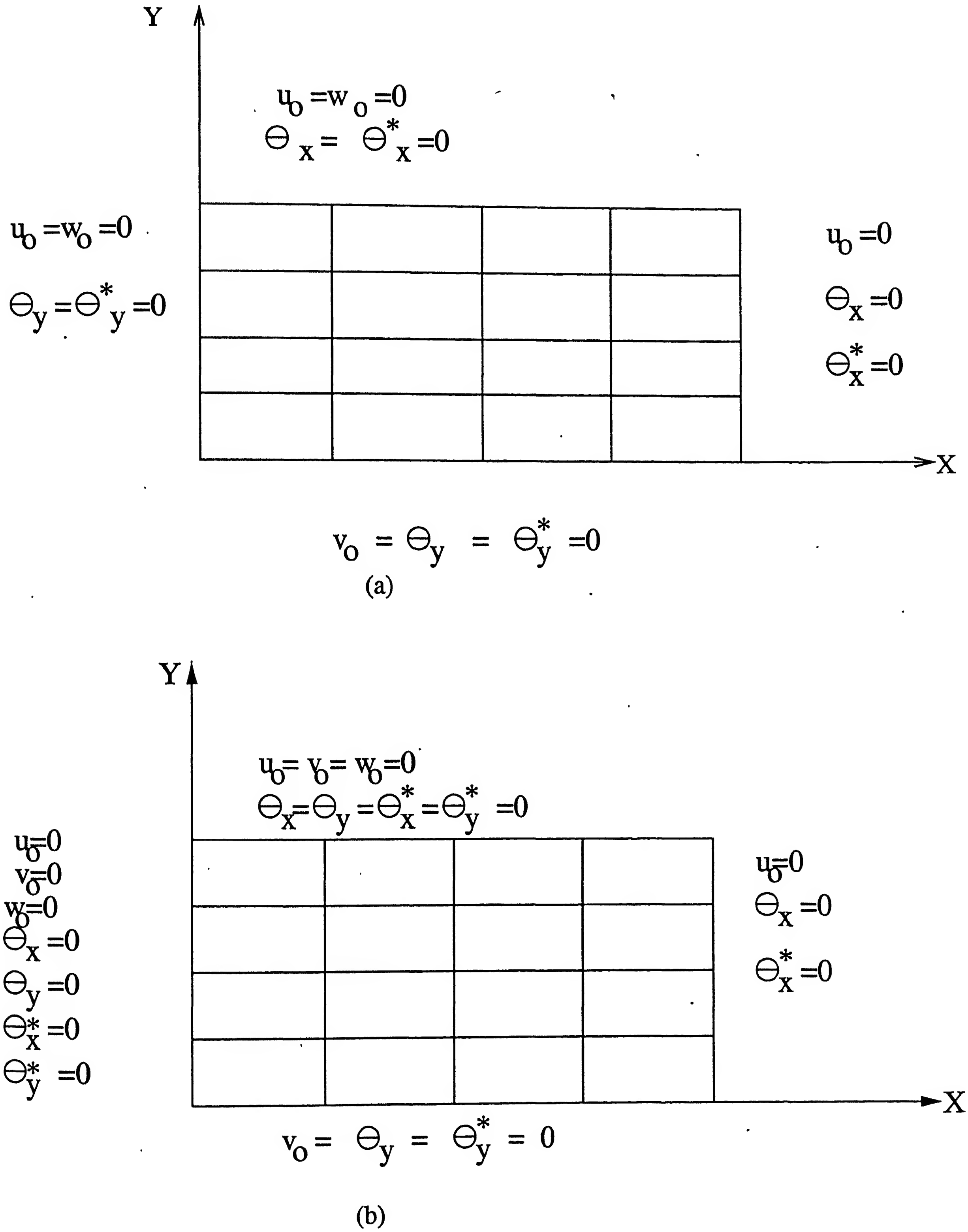

Figure 2.1: (a) Simply-supported boundary conditions. (b) Clamped B.C's.

2.4 Failure criteria

The criteria for the failure analysis of laminates, at macro level, which is ply level failure, can be classified into two groups, namely (i) independent failure criteria, and (ii) polynomial failure criteria. In independent failure criteria the interaction among different stresses and strains does not exist. Therefore, in general, the polynomial failure criteria proposed by Tsai-Wu (1971) is used. The tensor equation of these failure criteria is expressed as:

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i \geq 1 \tag{2.80}$$

where F_i and F_{ij} are the strength tensors of second and fourth rank, and σ_i (contracted stress notation is used) is the stress component in the material coordinate system. In a more explicit form, the criteria (2.80) is

$$\begin{aligned} F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 + 2F_{13}\sigma_1\sigma_3 + 2F_{23}\sigma_2\sigma_3 + F_{11}\sigma_1^2 \\ +F_{22}\sigma_2^2 + F_{33}\sigma_3^2 + F_{44}\sigma_4^2 + F_{55}\sigma_5^2 + F_{66}\sigma_6^2 + \ldots \geq 1 \end{aligned} \tag{2.81}$$

in which σ_1, σ_2 and σ_3 are the normal stress components and in three principal material directions, while σ_4, σ_5 and σ_6 are the shear stress components at a point in planes 2-3, 1-3 and 1-2 respectively.

Following are the various polynomial criteria used (Reddy and Pandey, 1987 and Reddy and Reddy, 1992,(see Ref.[36]) for details):

- Maximum Stress Criteria
- Maximum Strain Criteria
- Tsai-Hill Criteria
- Hoffman Criteria

- Tsai-Wu Criteria

Composite materials are made by mechanically very dissimilar phases; namely a stiff elastic brittle fibre and a compliant yielding matrix. The fibre strength is direction dependent; and it possesses greater strength in the longitudinal than in the transverse direction. Hence, the strength of a lamina along the fibre direction is mainly due to the fibre strength and transverse to the fibre direction is mainly due to the matrix. Due to difference in interlaminar stresses, fibre debonding is also a possibility. Consequently, laminates fail in variety of different modes, but all failure modes can not be represented by single failure envelope. For convenience, the failure modes are categorised into the following groups:

Fibre Mode: Tensile fibre mode is described by breakage and compressive fibre mode is described by fibre buckling.

Matrix Mode: Matrix mode of failure is described by the failure due to in-plane normal stress transverse to the fibre direction, in-plane shear stresses and transverse shear in the lamina.

Delamination Mode: It is produced by transverse interlaminar stresses or transverse normal and shear components in layers adjacent to the interface.

2.5 Definition of failure

2.5.1 Lamina/Ply failure

The ply failure is said to have occurred when the state of stress at any point within the lamina satisfies the selected failure criterion. The first-ply failure refers to the first instant at which one or more plies fail at the same load.

2.5.2 Onset of delamination

The following two criteria are employed to predict the onset of delamination:

- The maximum stress criterion
- Interlaminar failure criterion([36])

As for the maximum stress criterion, delamination at any interface is said to have occurred when any of the transverse stress components in any of the two layers adjacent to the interface becomes equal to, or greater than its corresponding strength. As for the interlaminar failure criterion, the onset of delamination takes place when the interlaminar transverse stress components satisfy the following expressions:

$$\left(\frac{\sigma_3}{\sigma_{DN}}\right)^2 + \frac{\tau_{13}^2 + \tau_{23}^2}{\sigma_{DS}^2} \geq 1 \tag{2.82}$$

where σ_3 is the transverse normal stress component; τ_{13}, τ_{23} are the transverse shear stress components in the principal material planes 1-3 and 2-3, respectively; σ_{DN} is the peel strength and σ_{DS} is the interlaminar shear strength; these are taken equal to the tensile normal transverse strength and transverse shear strength (corresponding to the plane 1-3) of a lamina, respectively.

Chapter 3

Finite Element Formulation

3.1 Finite element formulation and solution technique:

The standard finite element technique is followed, in which the total solution domain is descretized in 'NE' sub-domains (or elements) such that

$$\Pi = \sum_{e=1}^{NE} \Pi^e(d) \tag{3.1}$$

where Π and Π^e are the total potential of the system and element respectively. The element potential can be expressed in terms of internal strain energy U^e and the external work-done W^e for an element 'e' as:

$$\Pi^e(d) = U^e - W^e \tag{3.2}$$

in which d is the vector of unknown displacement variables in the problem and defined by equation (2.53). If the same interpolation functi on is used to define all the componenets of the generalised displacement vector d , we can write,

$$d = \sum_{i=1}^{nn} n_i * d_i \tag{3.3}$$

where , n_i is the interpolating (shape) function associated with node i ,d_i is the value of d

corresponding to the node i and nn is the no. of nodes in the element.

Let us say the degrees of freedom are given as:

$$u_0 = \sum d_i^1 * n_i^1 \tag{3.4}$$

$$v_0 = \sum d_i^2 * n_i^2 \tag{3.5}$$

$$w_0 = \sum d_i^3 * n_i^3 \tag{3.6}$$

$$\theta_x = \sum d_i^4 * n_i^4 \tag{3.7}$$

$$\theta_y = \sum d_i^5 * n_i^5 \tag{3.8}$$

$$\theta_x^* = \sum d_i^6 * n_i^6 \tag{3.9}$$

$$\theta_y^* = \sum d_i^7 * n_i^7 \tag{3.10}$$

Now, the variational energy is:

$$\delta\Pi^e(d) = \delta U^e - \delta W^e \tag{3.11}$$

$\Rightarrow$

$$\delta U^e(d) = \int_V \delta\epsilon_{ij}^T \cdot \sigma_{ij}\, dV \tag{3.12}$$

or,

$$\begin{aligned} \delta U^e(d) &= \int_A \int_{-h/2}^{h/2} (\sigma_{xx} \cdot \delta\epsilon^{xx} + \sigma_{yy} \cdot \delta\epsilon^{yy} + \tau_{xy} \cdot \delta\gamma^{xy} + \\ &\quad \tau_{yz} \cdot \delta\gamma^{yz} + \tau_{xz} \cdot \delta\gamma^{xz})\, dV \end{aligned} \tag{3.13}$$

Now, by using equations (2.23)-(2.27), (2.36)-(2.37),(2.57)-(2.58), and (3.4)-(3.10) in equation (3.13), we shall get the following stiffness coefficients corresponding to the particular

degree of freedom as:

$$K_{ij}^{(1)} = n_{i,x}^{(1)} * N_x(n_j) + n_{i,y}^{(1)} * N_{xy}(n_j) \tag{3.14}$$

$$K_{ij}^{(2)} = n_{i,y}^{(2)} * N_y(n_j) + n_{i,x}^{(2)} * N_{xy}(n_j) \tag{3.15}$$

$$\begin{aligned} K_{ij}^{(3)} &= n_{i,x}^{(3)} * (N_x(n_j) * w_{,x} + N_{xy}(n_j) * w_{,y} + Q_x(n_j)) \\ &\quad + n_{i,y}^{(3)} * (N_y(n_j) * w_{,y} + N_{xy}(n_j) * w_{,x} + Q_y(n_j)) \end{aligned} \tag{3.16}$$

$$K_{ij}^{(4)} = n_i^{(4)} * Q_x(n_j) + n_{i,x}^{(4)} * M_x(n_j) + n_{i_y}^{(4)} * M_{xy}(n_j) \tag{3.17}$$

$$K_{ij}^{(5)} = n_i^{(5)} * Q_y(n_j) + n_{i,y}^{(5)} * M_y(n_j) + n_{i_x}^{(5)} * M_{xy}(n_j) \tag{3.18}$$

$$K_{ij}^{(6)} = 3.0 * n_i^{(6)} * Q_x^*(n_j) + n_{i,x}^{(6)} * M_x^*(n_j) + n_{i,y}^{(6)} * M_{xy}^*(n_j) \tag{3.19}$$

$$K_{ij}^{(7)} = 3.0 * n_i^{(7)} * Q_y^*(n_j) + n_{i,y}^{(7)} * M_y^*(n_j) + n_{i,x}^{(7)} * M_{xy}^*(n_j) \tag{3.20}$$

where from equations (2.17)-(2.22),(2.60),(3.4)-(3.10), and (3.14) -(3.20),we have

$$\begin{aligned} N_x(n_j) &= (A_{11} * n_{j_x}^{(1)} + A_{13} * n_{j_y}^{(1)}) \\ &\quad + (A_{12} * n_{j,y}^{(2)} + A_{13} * n_{j,x}^{(2)}) \\ &\quad + (B_{11} * n_{j,x}^{(4)} + B_{13} * n_{j,y}^{(4)}) \\ &\quad + (B_{12} * n_{j,y}^{(5)} + B_{13} * n_{j,x}^{(5)}) \\ &\quad + (B_{14} * n_{j,x}^{(6)} + B_{16} * n_{j,y}^{(6)}) \\ &\quad + (B_{15} * n_{j,y}^{(7)} + B_{16} * n_{j,x}^{(7)}) \\ &\quad + (A_{11} * w_{,x} * n_{j,x}^{(3)} + A_{12} * w_{,y} * n_{j_y}^{(3)}) \\ &\quad + (A_{13} * w_{,x} * n_{j,y}^{(3)} + A_{13} * w_{,y} * n_{j,x}^{(3)}) \end{aligned} \tag{3.21}$$

$$\begin{aligned} N_y(n_j) &= (A_{12} * n_{j_x}^{(1)} + A_{23} * n_{j_y}^{(1)}) \\ &\quad + (A_{22} * n_{j,y}^{(2)} + A_{23} * n_{j,x}^{(2)}) \end{aligned}$$

$$
\begin{aligned}
&+(B_{12} * n_{j,x}^{(4)} + B_{23} * n_{j,y}^{(4)}) \\
&+(B_{22} * n_{j,y}^{(5)} + B_{23} * n_{j,x}^{(5)}) \\
&+(B_{24} * n_{j,x}^{(6)} + B_{26} * n_{j,y}^{(6)}) \\
&+(B_{25} * n_{j,y}^{(7)} + B_{26} * n_{j,x}^{(7)}) \\
&+(A_{12} * w_{,x} * n_{j,x}^{(3)} + A_{22} * w_{,y} * n_{jy}^{(3)}) \\
&+(A_{23} * w_{,x} * n_{j,y}^{(3)} + A_{23} * w_{,y} * n_{j,x}^{(3)})
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
N_{xy}(n_j) \;=\; &(A_{13} * n_{jx}^{(1)} + A_{33} * n_{jy}^{(1)}) \\
&+(A_{23} * n_{j,y}^{(2)} + A_{33} * n_{j,x}^{(2)}) \\
&+(B_{13} * n_{j,x}^{(4)} + B_{33} * n_{j,y}^{(4)}) \\
&+(B_{23} * n_{j,y}^{(5)} + B_{13} * n_{j,x}^{(5)}) \\
&+(B_{14} * n_{j,x}^{(6)} + B_{16} * n_{j,y}^{(6)}) \\
&+(B_{15} * n_{j,y}^{(7)} + B_{16} * n_{j,x}^{(7)}) \\
&+(A_{13} * w_{,x} * n_{j,x}^{(3)} + A_{23} * w_{,y} * n_{jy}^{(3)}) \\
&+(A_{33} * w_{,x} * n_{j,y}^{(3)} + A_{33} * w_{,y} * n_{j,x}^{(3)})
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
M_x(n_j) \;=\; &(B_{11} * n_{j,x}^{(1)} + B_{31} * n_{j,y}^{(1)}) \\
&+(B_{12} * n_{j,y}^{(2)} + B_{31} * n_{j,x}^{(2)}) \\
&+(D_{b_{11}} * n_{j,x}^{(4)} + D_{b_{13}} * n_{j,y}^{(4)}) \\
&+(D_{b_{12}} * n_{j,y}^{(5)} + D_{b_{13}} * n_{j,x}^{(5)}) \\
&+(D_{b_{14}} * n_{j,x}^{(6)} + D_{b_{16}} * n_{j,y}^{(6)}) \\
&+(D_{b_{15}} * n_{j,y}^{(7)} + D_{b_{16}} * n_{j,x}^{(7)})
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
M_y(n_j) \;=\; &(B_{12} * n_{j,x}^{(1)} + B_{32} * n_{j,y}^{(1)}) \\
&+(B_{22} * n_{j,y}^{(2)} + B_{32} * n_{j,x}^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&+(D_{b_{21}} * n_{j,x}^{(4)} + D_{b_{23}} * n_{j,y}^{(4)}) \\
&+(D_{b_{22}} * n_{j,y}^{(5)} + D_{b_{23}} * n_{j,x}^{(5)}) \\
&+(D_{b_{24}} * n_{j,x}^{(6)} + D_{b_{26}} * n_{j,y}^{(6)}) \\
&+(D_{b_{25}} * n_{j,y}^{(7)} + D_{b_{26}} * n_{j,x}^{(7)})
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
M_{xy}(n_j) \;=\; &(B_{13} * n_{j,x}^{(1)} + B_{33} * n_{j,y}^{(1)}) \\
&+(B_{23} * n_{j,y}^{(2)} + B_{33} * n_{j,x}^{(2)}) \\
&+(D_{b_{31}} * n_{j,x}^{(4)} + D_{b_{33}} * n_{j,y}^{(4)}) \\
&+(D_{b_{32}} * n_{j,y}^{(5)} + D_{b_{33}} * n_{j,x}^{(5)}) \\
&+(D_{b_{34}} * n_{j,x}^{(6)} + D_{b_{36}} * n_{j,y}^{(6)}) \\
&+(D_{b_{35}} * n_{j,y}^{(7)} + D_{b_{36}} * n_{j,x}^{(7)})
\end{aligned} \tag{3.26}
$$

$$
\begin{aligned}
M_x^*(n_j) \;=\; &(B_{14} * n_{j,x}^{(1)} + B_{34} * n_{j,y}^{(1)}) \\
&+(B_{24} * n_{j,y}^{(2)} + B_{34} * n_{j,x}^{(2)}) \\
&+(D_{b_{41}} * n_{j,x}^{(4)} + D_{b_{43}} * n_{j,y}^{(4)}) \\
&+(D_{b_{42}} * n_{j,y}^{(5)} + D_{b_{43}} * n_{j,x}^{(5)}) \\
&+(D_{b_{44}} * n_{j,x}^{(6)} + D_{b_{46}} * n_{j,y}^{(6)}) \\
&+(D_{b_{45}} * n_{j,y}^{(7)} + D_{b_{46}} * n_{j,x}^{(7)})
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
M_y^*(n_j) \;=\; &(B_{15} * n_{j,x}^{(1)} + B_{35} * n_{j,y}^{(1)}) \\
&+(B_{25} * n_{j,y}^{(2)} + B_{35} * n_{j,x}^{(2)}) \\
&+(D_{b_{51}} * n_{j,x}^{(4)} + D_{b_{53}} * n_{j,y}^{(4)}) \\
&+(D_{b_{52}} * n_{j,y}^{(5)} + D_{b_{53}} * n_{j,x}^{(5)}) \\
&+(D_{b_{54}} * n_{j,x}^{(6)} + D_{b_{56}} * n_{j,y}^{(6)}) \\
&+(D_{b_{55}} * n_{j,y}^{(7)} + D_{b_{56}} * n_{j,x}^{(7)})
\end{aligned} \tag{3.28}
$$

$$\begin{aligned} M^*_{xy}(n_j) &= (B_{16} * n^{(1)}_{j,x} + B_{36} * n^{(1)}_{j,y}) \\ &+(B_{26} * n^{(2)}_{j,y} + B_{36} * n^{(2)}_{j,x}) \\ &+(D_{b_{61}} * n^{(4)}_{j,x} + D_{b_{63}} * n^{(4)}_{j,y}) \\ &+(D_{b_{62}} * n^{(5)}_{j,y} + D_{b_{63}} * n^{(5)}_{j,x}) \\ &+(D_{b_{64}} * n^{(6)}_{j,x} + D_{b_{66}} * n^{(6)}_{j,y}) \\ &+(D_{b_{65}} * n^{(7)}_{j,y} + D_{b_{66}} * n^{(7)}_{j,x}) \end{aligned} \tag{3.29}$$

$$\begin{aligned} Q_x(n_j) &= (D_{s_{11}} * n^{(4)}_j + D_{s_{12}} * n^{(5)}_j) \\ &+(D_{s_{11}} * n^{(3)}_{j,x} + D_{s_{12}} * n^{(3)}_{j,y}) \end{aligned} \tag{3.30}$$

$$\begin{aligned} Q_y(n_j) &= (D_{s_{21}} * n^{(4)}_j + D_{s_{22}} * n^{(5)}_j) \\ &+(D_{s_{21}} * n^{(3)}_{j,x} + D_{s_{22}} * n^{(3)}_{j,y}) \end{aligned} \tag{3.31}$$

$$Q^*_x(n_j) = 3 \cdot D_{s_{33}} * n^{(6)}_j + 3 \cdot D_{s_{34}} * n^{(7)}_j) \tag{3.32}$$

$$Q^*_y(n_j) = 3 \cdot D_{s_{43}} * n^{(6)}_j + 3 \cdot D_{s_{44}} * n^{(7)}_j) \tag{3.33}$$

3.1.1 Equivalent nodal load vector:

The external work-done due to transeverse loads is expressed in terms of shape functions as:

$$W^{(e)} = \sum_{i=1}^{nn} W^{(e)}_i . n_i \tag{3.34}$$

where 'e' denotes the elemental value, and

$$W^{(e)}_i = \begin{bmatrix} 0 & 0 & q^{(e)} & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{3.35}$$

3.1.2 Non-linear equilibrium equation:

For equilibrium of the element, we have

$$\delta\Pi^{(e)} = 0 \tag{3.36}$$

i.e.,

$$\delta\left(U^{(e)} - W^{(e)}\right) = 0 \tag{3.37}$$

Equation (3.37) gives the non-linear equilibrium equations after collecting the coefficients with displacement term and loading term separately.

$$\Psi(d_i) = K_{ij}(\tilde{d}) \cdot d_j - F_i \tag{3.38}$$

where,

$$F_i = \int_A n_i I_7 \cdot \left\{ \; W_i^e \; \right\} \, dA \tag{3.39}$$

where, I_7 is the 7×7 unit matrix.

3.2 Solution to non-linear equilibrium equation:

The solution algorithms for the assembled non-linear equilibrium equations (3.38) are based on the Newton-Raphson method, which consists of a series of linear solutions.If an initial estimate d_i, for the total displacements, gives the residual(or unbalanced) forces $\Psi(d_i) \neq 0$,then an improved value d_{i+1} is obtained by equating to zero the linearized Taylor's series expansion of $\Psi(d_{i+1})$ in the neighbourhood of d_i as,

$$\Psi(d_{i+1}) \simeq \Psi(d_i) + K_T \cdot \Delta d_i = 0 \tag{3.40}$$

where K_T is known as assembled tangent stiffness matrix evaluated at d_I and is given by,

$$K_T = \left[\frac{\partial\Psi(d_i)}{\partial\{d\}}\right] \tag{3.41}$$

Equation (3.40)is the linear incremental equilibrium equation which gives the linearized approximation to the relation between the residualforces and incremental displacements, Δd_i, at a point d_i on the equilibrium path. The improved solution is then found as:

$$d_{i+1} = d_i + \Delta d_i \tag{3.42}$$

Equations(3.40) and (3.42) represent the Newton-Raphson solution approach to the nonlinear equations (3.38). The terms $\Psi(d_i)$ and Δd_i give the measures of the convergence of the solution. To improve the numerical stability and to give intermediate results, the load F_i is usually applied in steps.

3.3 Tangent stiffness matrix:

The tangent stiffness matrix given in equation (3.41) may be written as,

$$K_T = \begin{bmatrix} \frac{\partial\Psi_1}{\partial d_1} & \frac{\partial\Psi_1}{\partial d_2} & \cdots & \frac{\partial\Psi_1}{\partial d_m} \\ \frac{\partial\Psi_2}{\partial d_1} & \frac{\partial\Psi_2}{\partial d_2} & \cdots & \frac{\partial\Psi_2}{\partial d_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\Psi_m}{\partial d_1} & \frac{\partial\Psi_m}{\partial d_2} & \cdots & \frac{\partial\Psi_m}{\partial d_m} \end{bmatrix} \tag{3.43}$$

where m is the number of degrees of freedom per element.

From equ. (3.38) and (3.41), we have

$$\begin{aligned} K_T &= \frac{\partial\Psi(d_i)}{\partial d_j} \\ &= K_{ij}(\tilde{d}) + \frac{\partial K_{ij}(\tilde{d})}{\partial d_j} \cdot d_l - \frac{\partial F_i}{\partial d_j} \end{aligned} \tag{3.44}$$

Therefore,

$$K_T = K_{ij}(\tilde{d}) + \left(\frac{\partial K_{ij}(\tilde{d})}{\partial d_j}\right) \cdot d_l \tag{3.45}$$

[since $\frac{\partial F_i}{\partial d_j} = 0$ for conservative loads.]

In equation (3.45),the term

$$\frac{\partial(K_{ij}(\tilde{d}))}{\partial(d_j)} = \frac{\partial^L K_{ij} +^{NL} K_{ij}(\tilde{d})}{\partial(d_j)} \tag{3.46}$$

$\Rightarrow$

$$\frac{\partial(K_{ij}(\tilde{d}))}{\partial(d_j)} = \frac{\partial^{NL} K_{ij}(\tilde{d})}{\partial(d_j)} \tag{3.47}$$

Therefore, from equations (3.45) and (3.47), the tangent stiffness matrix becomes:

$$K_T = K_{ij}(\tilde{d}) + \left(\frac{\partial(^{NL}K_{ij}(\tilde{d}))}{\partial d_j}\right) \cdot d_l \tag{3.48}$$

We can find the non-linear part of stiffness matrix $K_{ij}^{NL}(\tilde{d})$ by synthesizing equations (3.14)-(3.20) and using equations (3.21)-(3.33) in these, so as:

$$\begin{aligned} {}^{NL}K_{ij}^{(1)} &= n_{i,x}^{(1)} \left\{ \begin{array}{c} A_{11}w_{,x}n_{j,x}^{(3)} + A_{12}w_{,y}n_{j,y}^{(3)} \\ +A_{13}w_{,x}n_{j,y}^{(3)} + A_{13}w_{,y}n_{j,x}^{3} \end{array} \right\} \\ &\quad +n_{i,y}^{(1)} \left\{ \begin{array}{c} A_{31}w_{,x}n_{j,x}^{(3)} + A_{32}w_{,y}n_{j,y}^{(3)} \\ +A_{33}w_{,x}n_{j,y}^{(3)} + A_{33}w_{,y}n_{j,x}^{3} \end{array} \right\} \end{aligned} \tag{3.49}$$

$$\begin{aligned} {}^{NL}K_{ij}^{(2)} &= n_{i,x}^{(2)} \left\{ \begin{array}{c} A_{21}w_{,x}n_{j,x}^{(3)} + A_{22}w_{,y}n_{j,y}^{(3)} \\ +A_{23}w_{,x}n_{j,y}^{(3)} + A_{23}w_{,y}n_{j,x}^{3} \end{array} \right\} \\ &\quad +n_{i,y}^{(2)} \left\{ \begin{array}{c} A_{31}w_{,x}n_{j,x}^{(3)} + A_{32}w_{,y}n_{j,y}^{(3)} \\ +A_{33}w_{,x}n_{j,y}^{(3)} + A_{33}w_{,y}n_{j,x}^{3} \end{array} \right\} \end{aligned} \tag{3.50}$$

$$\begin{aligned} {}^{NL}K_{ij}^{(3)} &= n_{i,x}^{(3)} \cdot \{ \\ &\quad (A_{11}w_{,x}n_{j,x}^{(1)} + A_{13}w_{,x}n_{j,y}^{(1)}) \\ &\quad +(A_{12}w_{,x}n_{j,y}^{(2)} + A_{13}w_{,x}n_{j,x}^{2}) \\ &\quad +(B_{11}w_{,x}n_{j,x}^{(4)} + B_{13}w_{,x}n_{j,y}^{(4)}) \end{aligned}$$

$$
\begin{aligned}
&+(B_{12}w_{,x}n^{(5)}_{j,y} + B_{13}w_{,x}n^{(5)}_{j,x}) \\
&+(B_{14}w_{,x}n^{(6)}_{j,x} + B_{16}w_{,x}n^{(6)}_{j,y}) \\
&+(B_{15}w_{,x}n^{(7)}_{j,y} + B_{16}w_{,x}n^{(7)}_{j,x}) \\
&+w_{i,x}\cdot [\\
&(A_{11}w_{,x}n^{(3)}_{j,x} + A_{12}w_{,y}n^{(3)}_{j,y}) \\
&+(A_{13}w_{,x}n^{(3)}_{j,y} + A_{13}w_{,y}n^{(3)}_{j,x})\ ] \\
&+(A_{31}w_{,y}n^{(1)}_{j,x} + A_{33}w_{,y}n^{(1)}_{j,y}) \\
&+(A_{32}w_{,y}n^{(2)}_{j,y} + A_{33}w_{,y}n^{(2)}_{j,x}) \\
&+(B_{31}w_{,y}n^{(4)}_{j,x} + B_{33}w_{,y}n^{(4)}_{j,y}) \\
&+(B_{32}w_{,y}n^{(5)}_{j,y} + B_{33}w_{,y}n^{(5)}_{j,x}) \\
&+(B_{34}w_{,y}n^{(6)}_{j,x} + B_{36}w_{,y}n^{(6)}_{j,y}) \\
&+(B_{35}w_{,y}n^{(7)}_{j,y} + B_{36}w_{,y}n^{(7)}_{j,x}) \\
&+w_{i,y}\cdot [\\
&(A_{31}w_{,x}n^{(3)}_{j,x} + A_{32}w_{,y}n^{(3)}_{j,y}) \\
&+(A_{33}w_{,x}n^{(3)}_{j,y} + A_{33}w_{,y}n^{(3)}_{j,x})\ ]\ \} \\
&+n^{(3)}_{i,y}\cdot \{ \\
&(A_{21}w_{,y}n^{(1)}_{j,x} + A_{23}w_{,y}n^{(1)}_{j,y}) \\
&+(A_{22}w_{,y}n^{(2)}_{j,y} + A_{23}w_{,y}n^{2}_{j,x}) \\
&+(B_{21}w_{,y}n^{(4)}_{j,x} + B_{23}w_{,y}n^{(4)}_{j,y}) \\
&+(B_{22}w_{,y}n^{(5)}_{j,y} + B_{23}w_{,y}n^{(5)}_{j,x}) \\
&+(B_{24}w_{,y}n^{(6)}_{j,x} + B_{26}w_{,y}n^{(6)}_{j,y}) \\
&+(B_{25}w_{,y}n^{(7)}_{j,y} + B_{26}w_{,y}n^{(7)}_{j,x}) \\
&+w_{i,y}\cdot [
\end{aligned}
\tag{3.51}
$$

$$(A_{21}w_{,x}n^{(3)}_{j,x} + A_{22}w_{,y}n^{(3)}_{j,y})$$
$$+(A_{23}w_{,x}n^{(3)}_{j,y} + A_{23}w_{,y}n^{(3)}_{j,x})\ ]$$
$$+(A_{31}w_{,x}n^{(1)}_{j,x} + A_{33}w_{,x}n^{(1)}_{j,y})$$
$$+(A_{32}w_{,x}n^{(2)}_{j,y} + A_{33}w_{,x}n^{(2)}_{j,x})$$
$$+(B_{31}w_{,x}n^{(4)}_{j,x} + B_{33}w_{,x}n^{(4)}_{j,y})$$
$$+(B_{32}w_{,x}n^{(5)}_{j,y} + B_{33}w_{,x}n^{(5)}_{j,x})$$
$$+(B_{34}w_{,x}n^{(6)}_{j,x} + B_{36}w_{,x}n^{(6)}_{j,y})$$
$$+(B_{35}w_{,x}n^{(7)}_{j,y} + B_{36}w_{,x}n^{(7)}_{j,x})$$
$$+w_{i,x}\cdot[$$
$$(A_{31}w_{,x}n^{(3)}_{j,x} + A_{32}w_{,y}n^{(3)}_{j,y})$$
$$+(A_{33}w_{,x}n^{(3)}_{j,y} + A_{33}w_{,y}n^{(3)}_{j,x})\ ]\ \}$$

$$\begin{aligned} {}^{NL}K^{(4)}_{ij} &= n^{(4)}_{i,x}\left\{\begin{array}{c} B_{11}w_{,x}n^{(3)}_{j,x} + B_{21}w_{,y}n^{(3)}_{j,y} \\ +B_{31}w_{,x}n^{(3)}_{j,y} + B_{31}w_{,y}n^{3}_{j,x} \end{array}\right\} \\ &\quad +n^{(4)}_{i,y}\left\{\begin{array}{c} B_{13}w_{,x}n^{(3)}_{j,x} + B_{23}w_{,y}n^{(3)}_{j,y} \\ +B_{33}w_{,x}n^{(3)}_{j,y} + B_{33}w_{,y}n^{3}_{j,x} \end{array}\right\} \end{aligned} \tag{3.52}$$

$$\begin{aligned} {}^{NL}K^{(5)}_{ij} &= n^{(5)}_{i,x}\left\{\begin{array}{c} B_{13}w_{,x}n^{(3)}_{j,x} + B_{23}w_{,y}n^{(3)}_{j,y} \\ +B_{33}w_{,x}n^{(3)}_{j,y} + B_{33}w_{,y}n^{3}_{j,x} \end{array}\right\} \\ &\quad +n^{(5)}_{i,y}\left\{\begin{array}{c} B_{12}w_{,x}n^{(3)}_{j,x} + B_{22}w_{,y}n^{(3)}_{j,y} \\ +B_{32}w_{,x}n^{(3)}_{j,y} + B_{32}w_{,y}n^{3}_{j,x} \end{array}\right\} \end{aligned} \tag{3.53}$$

$$\begin{aligned} {}^{NL}K^{(6)}_{ij} &= n^{(6)}_{i,x}\left\{\begin{array}{c} B_{14}w_{,x}n^{(3)}_{j,x} + B_{24}w_{,y}n^{(3)}_{j,y} \\ +B_{34}w_{,x}n^{(3)}_{j,y} + B_{34}w_{,y}n^{3}_{j,x} \end{array}\right\} \\ &\quad +n^{(6)}_{i,y}\left\{\begin{array}{c} B_{16}w_{,x}n^{(3)}_{j,x} + B_{26}w_{,y}n^{(3)}_{j,y} \\ +B_{36}w_{,x}n^{(3)}_{j,y} + B_{36}w_{,y}n^{3}_{j,x} \end{array}\right\} \end{aligned} \tag{3.54}$$

$$
\begin{aligned}
{}^{NL}K_{ij}^{(7)} &= n_{i,x}^{(7)} \left\{ \begin{array}{c} B_{16} w_{,x} n_{j,x}^{(3)} + B_{26} w_{,y} n_{j,y}^{(3)} \\ + B_{36} w_{,x} n_{j,y}^{(3)} + B_{36} w_{,y} n_{j,x}^{3} \end{array} \right\} \\
&\quad + n_{i,y}^{(7)} \left\{ \begin{array}{c} B_{15} w_{,x} n_{j,x}^{(3)} + B_{25} w_{,y} n_{j,y}^{(3)} \\ + B_{35} w_{,x} n_{j,y}^{(3)} + B_{35} w_{,y} n_{j,x}^{3} \end{array} \right\}
\end{aligned} \tag{3.55}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial^{NL} K_{il}}{\partial d_j^{(1)}} &= \frac{\partial^{NL} K_{il}}{\partial d_j^{(2)}} = \frac{\partial^{NL} K_{il}}{\partial d_j^{(4)}} \\
&= \frac{\partial^{NL} K_{il}}{\partial d_j^{(5)}} = \frac{\partial^{NL} K_{il}}{\partial d_j^{(6)}} = \frac{\partial^{NL} K_{il}}{\partial d_j^{(7)}}
\end{aligned} \tag{3.56}
$$

and,

$$
\begin{aligned}
\frac{\partial^{NL} K_{il}^{(1)}}{\partial d_j^{(3)}} &= n_{i,x}^{(1)} \left\{ \begin{array}{c} A_{11} n_{j,x}^{(3)} n_{l,x}^{(3)} + A_{12} n_{j,y}^{(3)} n_{l,y}^{(3)} \\ + A_{31} n_{j,x}^{(3)} n_{l,y}^{(3)} + A_{13} n_{j,y}^{(3)} n_{l,x}^{3} \end{array} \right\} \\
&\quad + n_{i,y}^{(1)} \left\{ \begin{array}{c} A_{31} n_{j,x}^{(3)} n_{l,x}^{(3)} + A_{32} n_{j,y}^{(3)} n_{l,y}^{(3)} \\ + A_{33} n_{j,x}^{(3)} n_{l,y}^{(3)} + A_{33} n_{j,y}^{(3)} n_{l,x}^{3} \end{array} \right\}
\end{aligned} \tag{3.57}
$$

or,

$$
\begin{aligned}
\frac{\partial^{NL} K_{il}^{(1)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} &= n_{i,x}^{(1)} \left\{ \begin{array}{c} A_{11} w_{,x} n_{j,x}^{(3)} + A_{12} w_{,y} n_{j,y}^{(3)} \\ + A_{13} w_{,x} n_{j,y}^{(3)} + A_{13} w_{,y} n_{j,x}^{3} \end{array} \right\} \\
&\quad + n_{i,y}^{(1)} \left\{ \begin{array}{c} A_{31} w_{,x} n_{j,x}^{(3)} + A_{32} w_{,y} n_{j,y}^{(3)} \\ + A_{33} w_{,x} n_{j,y}^{(3)} + A_{33} w_{,y} n_{j,x}^{3} \end{array} \right\}
\end{aligned} \tag{3.58}
$$

$\Rightarrow$

$$
\frac{\partial^{NL} K_{il}^{(1)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} =^{NL} K_{ij}^{(1)} \tag{3.59}
$$

Similarly,

$$
\frac{\partial^{NL} K_{il}^{(2)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} =^{NL} K_{ij}^{(2)} \tag{3.60}
$$

$$
\frac{\partial^{NL} K_{il}^{(4)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} =^{NL} K_{ij}^{(4)} \tag{3.61}
$$

$$\frac{\partial^{NL} K_{il}^{(5)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} =^{NL} K_{ij}^{(5)} \tag{3.62}$$

$$\frac{\partial^{NL} K_{il}^{(6)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} =^{NL} K_{ij}^{(6)} \tag{3.63}$$

$$\frac{\partial^{NL} K_{il}^{(7)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} =^{NL} K_{ij}^{(7)} \tag{3.64}$$

and,

$$\begin{aligned}
\frac{\partial^{NL} K_{il}^{(3)}}{\partial d_j^{(3)}} &= n_{i,x}^{(3)} \left(N_x(n_l) \cdot n_{l,x}^{(3)} + N_{xy}(n_l) \cdot n_{l,y}^{(3)} \right) \\
&+ n_{i,y}^{(3)} \left(N_y(n_l) \cdot n_{l,y}^{(3)} + N_{xy}(n_l) \cdot n_{l,x}^{(3)} \right) \\
&+ n_{i,x}^{(3)} \left(\frac{\partial N_x(n_l)}{\partial d_j^{(3)}} \cdot w_{,x} + \frac{\partial N_{xy}(n_l)}{\partial d_j^{(3)}} \cdot w_{,y} \right) \\
&+ n_{i,y}^{(3)} \left(\frac{\partial N_y(n_l)}{\partial d_j^{(3)}} \cdot w_{,y} + \frac{\partial N_{xy}(n_l)}{\partial d_j^{(3)}} \cdot w_{,x} \right)
\end{aligned} \tag{3.65}$$

Therefore,

$$\begin{aligned}
\frac{\partial^{NL} K_{il}^{(3)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} &= n_{i,x}^{(3)} \left(\begin{array}{c} N_x(n_l) \cdot w_{,x} + N_{xy}(n_l) \cdot w_{,y}) \\ + n_{i,y}^{(3)} (N_y(n_l) \cdot w_{l,y} + N_{xy}(n_l) \cdot w_{,x} \end{array} \right) \\
&+ n_{i,x}^{(3)} \left(\frac{\partial N_x(n_l)}{\partial d_j^{(3)}} \cdot w_{,x} + \frac{\partial N_{xy}(n_l)}{\partial d_j^{(3)}} \cdot w_{,y} \right) \\
&+ n_{i,y}^{(3)} \left(\frac{\partial N_y(n_l)}{\partial d_j^{(3)}} \cdot w_{,y} + \frac{\partial N_{xy}(n_l)}{\partial d_j^{(3)}} \cdot w_{,x} \right)
\end{aligned} \tag{3.66}$$

From equation (3.14)-(3.20),we have

$$\begin{aligned}
{}^{NL} K_{ij}^{(3)} &= n_{i,x}^{(3)} \left(N_x(n_j) \cdot w_{,x} + N_{xy}(n_j) \cdot w_{,y} \right) \\
&+ n_{i,y}^{(3)} \left(N_y(n_j) \cdot w_{,y} + N_{xy}(n_j) \cdot w_{,x} \right)
\end{aligned} \tag{3.67}$$

and from equation(3.21)-(3.33), we have

$$\begin{aligned}
\frac{\partial N_x(n_l)}{\partial d_j^{(3)}} &= \frac{1}{2} \left(A_{11} n_{j,x}^{(3)} \cdot nl_{,x}{}^{(3)} + A_{12} n_{j,y}^{(3)} \cdot n_{l,y}^{(3)} \right) \\
&+ \frac{1}{2} \left(A_{13} n_{j,x}^{(3)} \cdot nl_{,y}{}^{(3)} + A_{13} n_{j,y}^{(3)} \cdot n_{l,x}^{(3)} \right)
\end{aligned} \tag{3.68}$$

$$\frac{\partial N_y(n_l)}{\partial d_j^{(3)}} = \frac{1}{2}\left(A_{21}n_{j,x}^{(3)} \cdot nl_{,x}{}^{(3)} + A_{22}n_{j,y}^{(3)} \cdot n_{l,y}^{(3)} \right) + \frac{1}{2}\left(A_{23}n_{j,x}^{(3)} \cdot nl_{,y}{}^{(3)} + A_{23}n_{j,y}^{(3)} \cdot n_{l,x}^{(3)} \right) \tag{3.69}$$

$$\frac{\partial N_{xy}(n_l)}{\partial d_j^{(3)}} = \frac{1}{2}\left(A_{31}n_{j,x}^{(3)} \cdot nl_{,x}{}^{(3)} + A_{32}n_{j,y}^{(3)} \cdot n_{l,y}^{(3)} \right) + \frac{1}{2}\left(A_{33}n_{j,x}^{(3)} \cdot nl_{,y}{}^{(3)} + A_{33}n_{j,y}^{(3)} \cdot n_{l,x}^{(3)} \right) \tag{3.70}$$

Thus,

$$\begin{aligned}
\frac{\partial^{NL}K_{il}^{(3)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} &= {}^{NL}K_{ij}^{(3)} \\
&+ \frac{1}{2} \cdot n_{i,x}^{(3)} \cdot w_{,x} \left\{ \begin{array}{c} A_{11}w_{,x}n_{j,x}^{(3)} + A_{12}w_{,y}n_{j,y}^{(3)} \\ +A_{13}w_{,x}n_{j,y}^{(3)} + A_{13}w_{,y}n_{j,x}^{3} \end{array} \right\} \\
&+ \frac{1}{2} \cdot n_{i,x}^{(3)} \cdot w_{,y} \left\{ \begin{array}{c} A_{31}w_{,x}n_{j,x}^{(3)} + A_{32}w_{,y}n_{j,y}^{(3)} \\ +A_{33}w_{,x}n_{j,y}^{(3)} + A_{33}w_{,y}n_{j,x}^{3} \end{array} \right\} \\
&+ \frac{1}{2} \cdot n_{i,y}^{(3)} \cdot w_{,y} \left\{ \begin{array}{c} A_{21}w_{,x}n_{j,x}^{(3)} + A_{22}w_{,y}n_{j,y}^{(3)} \\ +A_{23}w_{,x}n_{j,y}^{(3)} + A_{23}w_{,y}n_{j,x}^{3} \end{array} \right\} \\
&+ \frac{1}{2} \cdot n_{i,y}^{(3)} \cdot w_{,x} \left\{ \begin{array}{c} A_{31}w_{,x}n_{j,x}^{(3)} + A_{32}w_{,y}n_{j,y}^{(3)} \\ +A_{33}w_{,x}n_{j,y}^{(3)} + A_{33}w_{,y}n_{j,x}^{(3)} \end{array} \right\}
\end{aligned} \tag{3.71}$$

or,

$$\begin{aligned}
\frac{\partial^{NL}K_{il}^{(3)}}{\partial d_j^{(3)}} \cdot d_l^{(3)} &= {}^{NL}K_{ij}^{(3)} + \left[n_{i,x}^{(3)} \cdot \left\{ w_{,x} \cdot^{NL} N_x(n_j) + w_{,y} \cdot^{NL} N_{xy}(n_j) \right\} \right] \\
&+ \left[n_{i,y}^{(3)} \cdot \left\{ w_{,x} \cdot^{NL} N_{xy}(n_j) + w_{,y} \cdot^{NL} N_y(n_j) \right\} \right]
\end{aligned} \tag{3.72}$$

where,

$$^{NL}N_x(n_j) = \frac{\partial N_x(n_l)}{\partial d_j^{(3)}} \cdot d_l^{(3)} \tag{3.73}$$

$$^{NL}N_y(n_j) = \frac{\partial N_y(n_l)}{\partial d_j^{(3)}} \cdot d_l^{(3)} \tag{3.74}$$

$$^{NL}N_{xy}(n_j) = \frac{\partial N_{xy}(n_l)}{\partial d_j^{(3)}} \cdot d_l^{(3)} \tag{3.75}$$

Therefore, from equation (3.48) and (3.59), we get

$$K_{T_{ij}}^{(1)} = K_{ij}^{(1)}(\tilde{d}) + {}^{NL}K_{ij}^{(1)} \tag{3.76}$$

From equations (3.48) and (3.60), we get

$$K_{T_{ij}}^{(2)} = K_{ij}^{(2)}(\tilde{d}) + {}^{NL}K_{ij}^{(2)} \tag{3.77}$$

From equations (3.48) and (3.71), we get

$$\begin{aligned} K_{T_{ij}}^{(3)} = \quad & K_{ij}^{(3)}(\tilde{d}) + {}^{NL}K_{ij}^{(3)} \\ & +[n_{i,x}^{(3)} \cdot \{w_{,x} \cdot^{NL} N_x(n_j) + w_{,y} \cdot^{NL} N_{xy}(n_j)\}] \\ & +[n_{i,y}^{(3)} \cdot \{w_{,x} \cdot^{NL} N_{xy}(n_j) + w_{,y} \cdot^{NL} N_y(n_j)\}] \end{aligned} \tag{3.78}$$

From equations (3.48) and (3.61), we get

$$K_{T_{ij}}^{(4)} = K_{ij}^{(4)}(\tilde{d}) + {}^{NL}K_{ij}^{(4)} \tag{3.79}$$

From equations (3.48) and (3.62), we get

$$K_{T_{ij}}^{(5)} = K_{ij}^{(5)}(\tilde{d}) + {}^{NL}K_{ij}^{(5)} \tag{3.80}$$

From equations (3.48) and (3.63), we get

$$K_{T_{ij}}^{(6)} = K_{ij}^{(6)}(\tilde{d}) + {}^{NL}K_{ij}^{(6)} \tag{3.81}$$

From equations (3.48) and (3.64), we get

$$K_{T_{ij}}^{(7)} = K_{ij}^{(7)}(\tilde{d}) + {}^{NL}K_{ij}^{(7)} \tag{3.82}$$

Now, by using equations (3.76)-(3.82), we can calculate the tangent stiffness matrix $[K_T]$.

The iterative solution procedure used in equations (3.38) and (3.40) is checked for convergence using the following criteria:

$$\left[\frac{\Psi^T \cdot \Psi}{F^T \cdot F}\right]^{1/2} \times 100 \leq \beta \tag{3.83}$$

where, β is sufficiently small ($\simeq 10^{-3}$).

Knowing the displacement vector$\{d\}$, we shall be able to calculate strains for the given laminate.Now, by using constitutive equations, we can get stresses at the certain point in the laminate.

3.4 Material non-linear solution technique:

We shall solve material non-linear problem by applying strains in increments. For the first value of strain, we shall calculate the various tangent modulii, i.e.,$E_T^{\prime s}$. Now we shall use these $E_T^{\prime s}$ to update the tangent stiffness matrix $[K_T]$ and then finding the displacement vector d. For these displacements, again calculating the strains through strain-displacement relations.Using this new set of strains we shall again calculate the new values of $E_T^{\prime s}$ and again update the tangent stiffness matrix to get new set of displacements. This procedure will be repeated till the following criterion is not satisfied:

$$\epsilon_{i+1} - \epsilon_i \leq \alpha \tag{3.84}$$

where, ϵ_{i+1} and ϵ_i are two consecutive values of strains during iterative procedure, and, α is very small $\simeq 10^{-6}$. Now, we shall go for the second value of the strains and shall again be repeating the whole procedure to find the displacement field which satisfies equation (3.84).

The whole procedure is repeated again for all other values of strain-vector. Thus, we shall be able to find the displacements upto desired accuracy.

Chapter 4

Results and discussion

4.1 Convergence Study

The material properties for the different materials are shown in table 4.1.These material constants are taken from the [24] & [27]. For the material nonlinear problem, we have taken the following material properties for the material A(table 4.1) as [27]:

$E_{p_{11}}$=5.0×10^6psi, $E_{p_{22}}$=2.0×10^6psi,

λ_{66}=2.0, λ_{44}=2.0, λ_{55}=2.0,

$\sigma_{p_{11}}$=0.005×10^6psi, $\sigma_{p_{22}}$=0.005×10^6psi,

m=2.0

Table 4.1: Material constants of different materials.[24] & [27]

Material	E_{11} ×10^6	E_{22} ×10^6	G_{12} ×10^6	G_{23} ×10^6	G_{13} ×10^6	ν_{12}	ν_{21}
A	25.0	1.0	0.5	0.2	0.5	0.25	0.01
B	30.0	30.0	11.4	11.4	11.4	0.316	0.316
C	1.8282	1.8315	0.3125	0.3125	0.3125	0.23949	0.24078
D	3.0	1.28	0.37	0.37	0.37	0.32	0.24

Table 4.2: Center Deflections of rectangular laminate using linear HSDT for different mesh

Mesh	Stacking Sequence	Simply-Supported	Clamped
2×2	0^0	1.715807×10^{-1}	5.83942×10^{-2}
	$0^0/90^0$	5.644679×10^{-1}	1.670752×10^{-1}
	$0^0/90^0/90^0/0^0$	4.937523×10^{-1}	1.488710×10^{-1}
	$[45^0/-45^0]_s$	4.809534×10^{-1}	3.73201×10^{-1}
4×4	0^0	1.464109×10^{-1}	5.6465×10^{-2}
	$0^0/90^0$	5.63067×10^{-1}	1.461067×10^{-1}
	$0^0/90^0/90^0/0^0$	4.916800×10^{-1}	1.42304×10^{-1}
	$[45^0/-45^0]_s$	3.292528×10^{-1}	2.16222×10^{-1}
8×8	0^0	1.460854×10^{-1}	5.6167×10^{-2}
	$0^0/90^0$	5.59695×10^{-1}	1.426537×10^{-1}
	$0^0/90^0/90^0/0^0$	4.889124×10^{-1}	1.40254×10^{-1}
	$[45^0/-45^0]_s$	3.265564×10^{-1}	2.12656×10^{-1}

The confidence in the finite element formulation is checked with the mesh refinements.Results are taken for the analysis with linear HSDT for various type of laminates of material 'A'(Table 4.1). It is found that 4x4 mesh gives equally good results as compared to those found in literature. But, we have used 2x2 mesh for various calculations, as this mesh size gives results of admissible accuracy and less computer time, when compared with 4x4 mesh and 8x8 mesh. Though the convergence of the values are achieved through the highly refined mesh, but the computational time will be too much. Therefore, we have settled with 2x2 mesh only.The results of convergence-study for central deflection of the simply-supported and clamped plate of material'A' are shown in the table 4.2. These results show that 8x8 mesh is quite accurate.

4.2 Various non-linear cases for cross ply and angle ply laminate

The geometric non-linear results are taken for the cross-ply with material (C) and for angle-ply laminates with material (A) and results are compared with that of [27]. The dimensions of the plate are $a = b = 12in.$ and thickness $h = 0.096in.$ In figure 4.1, the load-deflection curve is drawn for 3-layer cross-ply laminate.Results are compared with those in [29], and, it is found that present study gives good results. The comparative study of 4-later cross-ply and 2-layer angle-ply laminates is shown in Figures Fig.4.2 and Fig.4.3.

Following results can be inferred from the curves:

i)Cross-ply laminate exhibits more non-linearity for same load and same dimensions when compared with angle-ply laminate.

ii)Increasing the no. of layers reduces the effect of non-linearity.

iii)Combined non-linear results can be best for higher loadings.

4.3 Orthotropic plate: Load vs Deflection for a=b=12in and thickness h=0.138in for material(D)

Results for the material nonlinear case are also presented for simply-supported $[0^o/90^o/0^o]$ orthotropic laminate in Fig.4.4 for a/h = 125 . Results for orthotropic plate are also taken for simply supported boundary conditions with various load values. Comparisons are made with values given in literature [24] and also with experimental values found in nonlinear analysis. Results are shown in Fig.4.4 and also compared with those in [24].

Some more results are shown in tables 4.10 and 4.11 respectively for a/h vs. central deflection for simply-supported, square, with material 'A', 3-layer and 2-layer cross-ply plate. Results are also compared with those in [8] and [27], and further ensure the reliability of present formulation.

4.4 Cross-ply laminates: Variation of deflections with a/h for various HSDT models

Results are taken for cross-ply laminates to find the variation of bending response with the side-to-thickness ratio, by using different combinations of non-linearities and also for linear HSDT analysis.It is found that for thicker laminates any of the HSDT can be used, but for thin laminates only non-linear HSDT can give admissible results.Results are shown for parametric study in tables 4.6, 4.7, 4.8 and 4.9 for a/h = 5, 10, 20 and 100 respectively and for different loadings.It is found that by using combined nonlinear analysis the results are slightly more conserved than by using only material nonlinear analysis.

Results for istropic plate are also taken for clamped boundary conditions with various load values. Comparisons are made with values given in literature and also with exact values.Results are tabulated in table 4.5.

4.5 Cross-ply laminates: Deflections with polynomial stress-strain relations for normal stresses

The polynomial stress-strain relations can be used in place of modified Ramberg-Osgood relations for the nomal stresses too. As the laminates are seldom encountered with the large stresses therefore we can use these polynomial equations for finding the results under the particular value of strains.Results are compared with those found with modified Ramberg-Osgood relations[27] and show equally good results. For comparison, we can see tables 4.5 and 4.7 simultaneously.

Table 4.3: Material Properties of T300/5208 (pre-peg) graphite-epoxy. [36]

Mechanical properties	Values	Strength properties	Values
E_1	132.58GPa	X_t	1.515GPa
E_2	10.8GPa	X_c	1.697GPa
E_3	10.8GPa	$Y_t = Z_t$	43.8MPa
$G_{12} = G_{13}$	5.7GPa	$Y_c = Z_c$	43.8MPa
$\nu_{12} = \nu_{13}$	0.24	R(shear strength in 2-3 plane)	67.6MPa
ν_{23}	0.49	S(shear strength in 1-3 plane) =T(sh. str. in 1-2 plane)	86.9MPa

4.6 Failure analysis by using linear HSDT,geometric non-linear HSDT and combined non-linear HSDT.

Results of failure analysis are found for transverse loads only.Results are taken for the various lamination schemes and for various combinations of non-linearities. Results are compared with those in [36]. First ply failure loads of the given laminates under the transverse load for the maximum strain criteria, are presented in Table(4.12), respectively for various theories, viz., linear, geometric nonlinear, and geometric as well as material nonlinear combined theory. It is known that maximum strain criterion yields the lowest strength for all load combinations. Therefore, maximum strain criterion is used to obtain the results witin lower bounds. Results are compared with those found in [36].

In this analysis, the laminate dimensions are taken as 479.88 mm X 479.88 mm X 3.72 mm with ply thickness as 0.2325mm([36]). The material properties are shown in table 4.3. The laminate is descretized into 4X4 meshes.

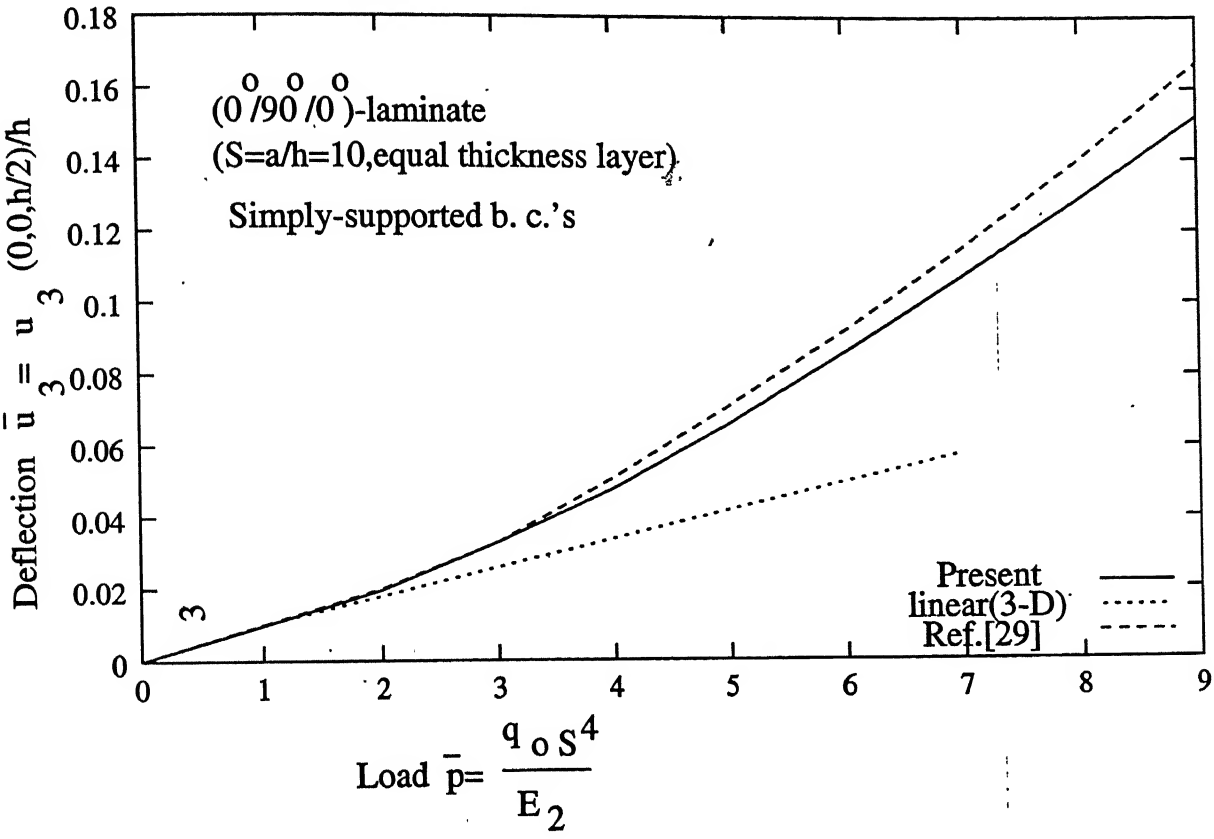

Figure 4.1: Load-deflection curve for Material nonlinear case for $a/h = 10$.

Table 4.4: Center Deflection for clamped, isotropic square plate for material (B),(a=b=300in, h=3in, $Q = (q_0 a^4)/(Eh^4)$)

Load Q	Exact [37]	Ref [37]	Present (linear HSDT)
17.79	0.237	0.2368	0.2417
38.30	0.471	0.4699	0.4723
63.40	0.695	0.6915	0.6961
95.00	0.912	0.9029	0.9127
134.9	1.121	1.1063	1.1217
184.0	1.323	1.3009	1.3243
245.0	1.521	1.4928	1.5311
318.0	1.714	1.6786	1.7160
402.0	1.902	1.8555	1.9044

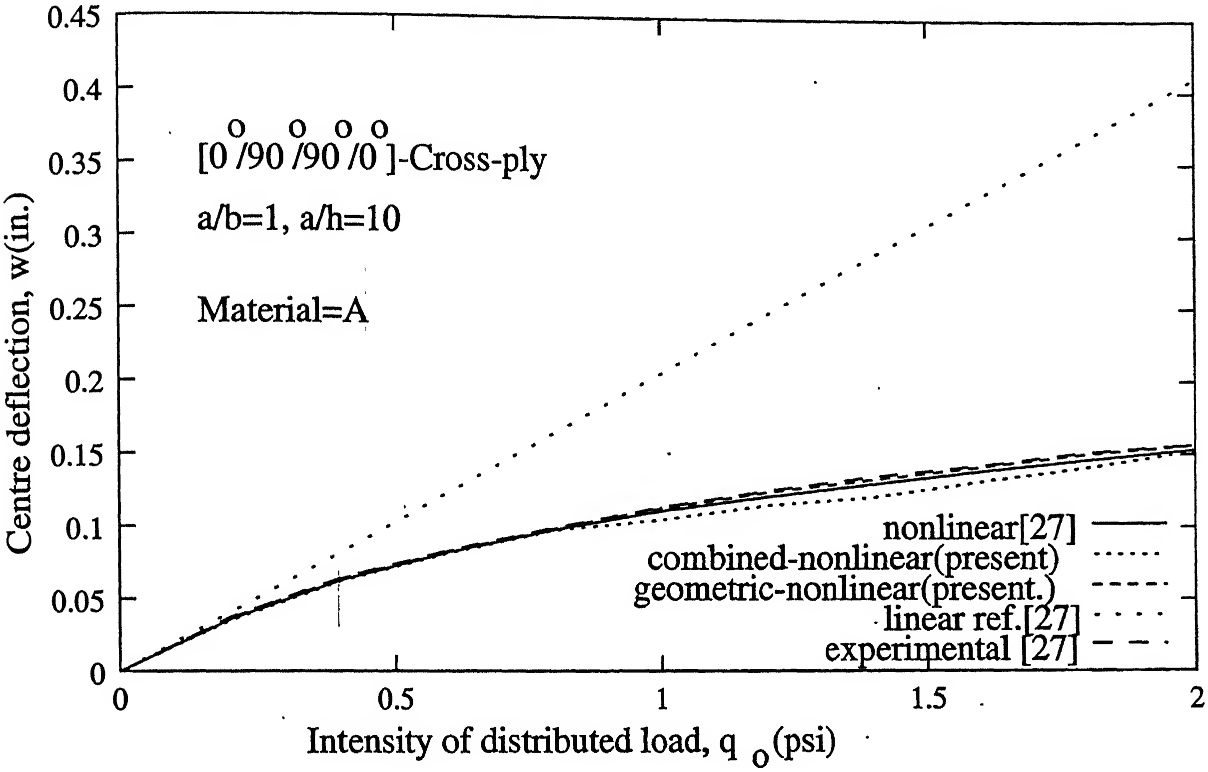

Figure 4.2: load vs.deflection for cross-ply plate.

Table 4.5: Cross-ply:Center Deflections with polynomial stress-strain model (cubic) for normal stress for material(A), for $a/h = 10$.

	Load Intensity	20	40	100	500	1000
L	S. S.($\times 10^3$)	2.14996	4.29992	10.7498	53.7490	107.49920
L	Clamped($\times 10^3$)	1.26759	2.5351	6.337	31.6850	63.3700
GNL	S. S.($\times 10^3$)	2.149957	4.29987	10.7489	41.2246	69.6578
GNL	Clamped($\times 10^3$)	1.267578	2.53504	6.3356	26.4623	42.6645
MNL	S. S.($\times 10^3$)	2.14802	4.298014	10.744	44.5612	75.8745
MNL	Clamped($\times 10^3$)	1.26712	2.53483	6.33568	28.5585	44.6792
GMNL	S. S.($\times 10^3$)	2.147864	4.297966	10.73895	43.4532	72.6547
GMNL	Clamped($\times 10^3$)	1.26687	2.52475	6.3153	27.9735	43.2375

(where, L=linear, GNL=geometric non-linear, MNL=material non-linear, GMNL=geometric and material,combined nonlinear.)

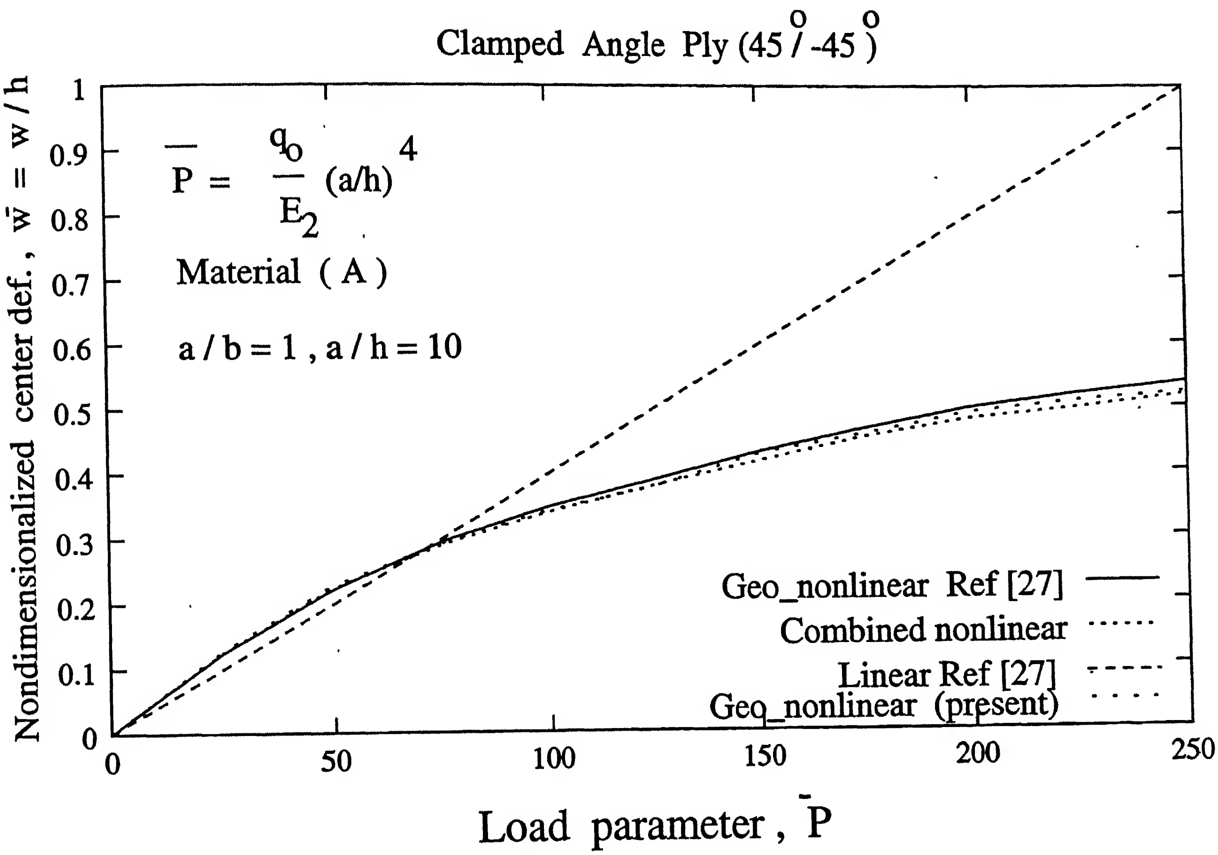

Figure 4.3: load vs.deflection for angle ply laminate.

Table 4.6: For $a/h = 5$

	Load Intensity	20	40	100	500	1000
L	S. S.($\times 10^4$)	3.034289	6.06857	15.17144	75.85720	151.71450
L	Clamped($\times 10^4$)	2.3310	4.6621	11.1655	55.8275	111.65520
GNL	S. S. ($\times 10^4$)	3.0342886	6.068573	15.17136	56.4355	92.8843
GNL	Clamped($\times 10^4$)	2.331075	4.662136	11.16550	47.9756	78.5465
MNL	S.S($\times 10^4$)	3.034288	6.068575	15.17139	59.7865	95.2374
MNL	Clamped($\times 10^4$)	2.331073	4.662121	11.16548	49.5575	81.6534
GMNL	S. S.($\times 10^4$)	3.034288	6.068574	15.171375	57.6785	94.2134
GMNL	Clamped($\times 10^4$)	2.331076	4.662142	11.61519	48.5764	79.8745

(where, L=linear, GNL=geometric non-linear, MNL=material non-linear,

GMNL=geometric and material,combined nonlinear.)

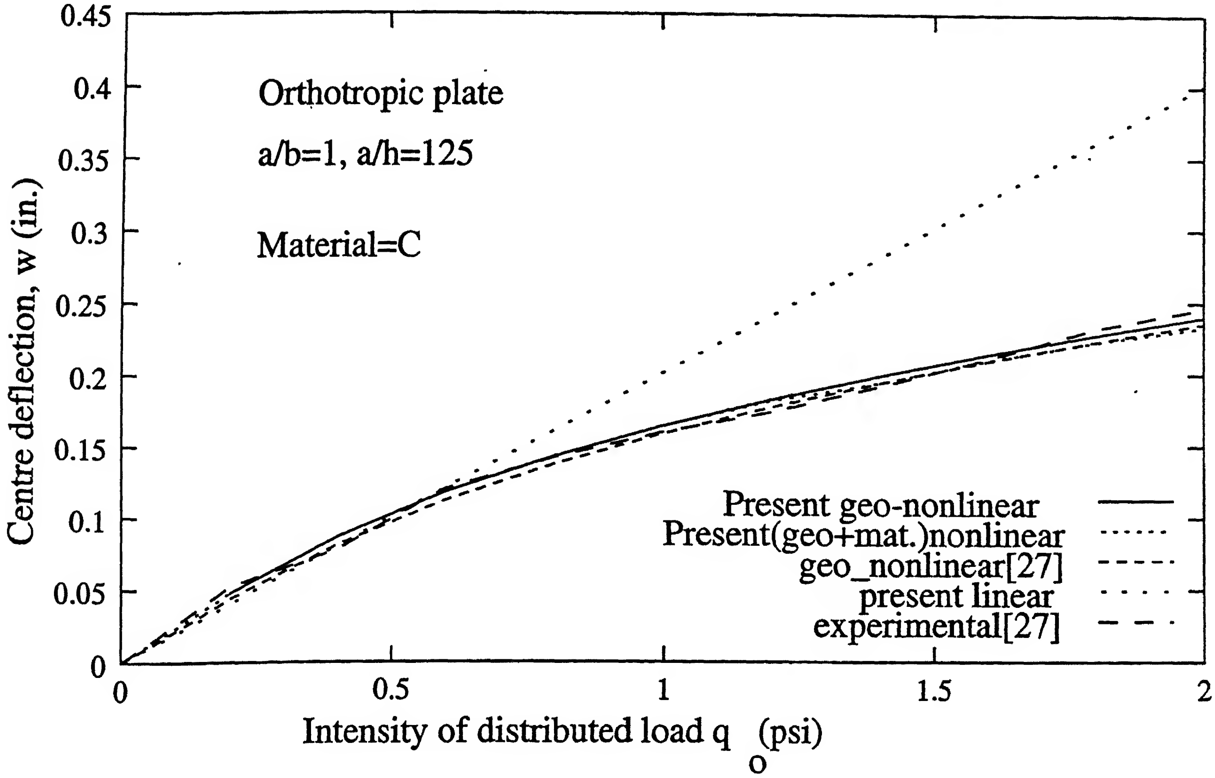

Figure 4.4: load vs.deflection for orthotropic plate.

Table 4.7: For $a/h = 10$

	Load Intensity	20	40	100	500	1000
L	S. S.($\times 10^3$)	2.14996	4.29992	10.7498	53.7490	107.49820
L	Clamped($\times 10^3$)	1.26759	2.5351	6.337	31.6855	63.3720
GNL	S. S.($\times 10^3$)	2.149957	4.29987	10.7489	44.8372	66.4735
GNL	Clamped($\times 10^3$)	1.267578	2.53504	6.3356	23.1563	39.8745
MNL	S. S.($\times 10^3$)	2.14992	4.299612	10.744	47.5637	68.76540
MNL	Clamped($\times 10^3$)	1.26758	2.53506	6.33598	26.7635	43.5675
GMNL	S. S.($\times 10^3$)	2.149972	4.299998	10.75093	45.7865	67.9725
GMNL	Clamped($\times 10^3$)	1.26757	2.53503	6.3355	24.8747	41.9736

(where, L=linear, GNL=geometric non-linear, MNL=material non-linear, GMNL=geometric and material,combined nonlinear.)

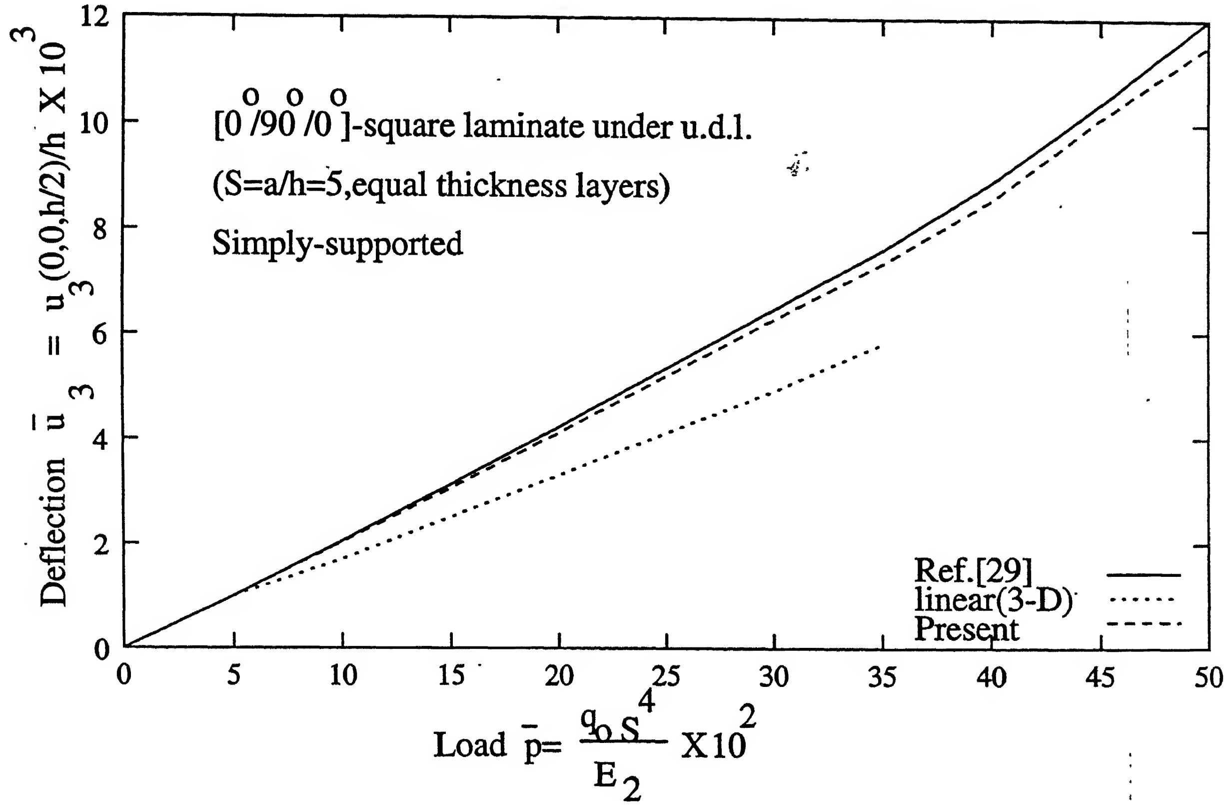

Figure 4.5: Load-deflection curve for Material nonlinear case for $a/h = 5$.

Table 4.8: For $a/h = 20$

	Load Intensity	20	40	100	500	1000
L	S. S. ($\times 10^2$)	2.4316	4.8632	12.1580	60.7902	121.58020
L	Clamped ($\times 10^1$)	8.2784	16.5569	41.3924	206.9620	413.9240
GNL	S. S. ($\times 10^2$)	2.43147	4.86209	10.1403	45.3213	70.8735
GNL	Clamped ($\times 10^1$)	8.2771	16.5464	38.2288	168.9427	288.3465
MNL	S. S. ($\times 10^2$)	2.431469	4.86205	10.1399	49.8745	73.5467
MNL	Clamped ($\times 10^1$)	8.27746	16.5487	38.2656	173.9473	294.3475
GMNL	S. S. ($\times 10^2$)	2.431473	4.862093	10.14032	47.1392	71.9685
GMNL	Clamped ($\times 10^1$)	8.27703	16.5452	38.2113	171.7438	291.7385

(where, L=linear, GNL=geometric non-linear, MNL=material non-linear, GMNL=geometric and material,combined nonlinear.)

Table 4.9: For $a/h = 100$

	Load Intensity	20	40	100	500	1000
L	S. S.	13.41730	26.83461	67.086353	335.431765	670.86353
L	Clamped	3.20981	6.41962	16.04905	80.24525	160.49050
GNL	S. S.	4.706729	6.317180	8.994607	8.999920	8.999945
GNL	Clamped	1.23547056	1.65246093	2.35682114	2.368799	2.37 23522
MNL	S. S.	4.78049241	6.45674941	9.2899640	9.355345	9.3702347
MNL	Clamped	1.273988	1.745240	2.6299976	2.6498564	2.661042
GMNL	S. S.	4.67617653	6.26272171	8.8931193	9.116382	9.286639
GMNL	Clamped	1.21966965	1.618488	2.276312	2.5693754	2.628495

(where, L=linear, GNL=geometric non-linear, MNL=material non-linear, GMNL=geometric and material,combined nonlinear.)

Table 4.10: Nondimensionalised center deflections for $(0^o/90^o/0^o)$ cross-ply S.S. square plate under UDL ($q_0 = 1.0; w = \frac{wE_2h^3 10^2}{q_0 a^4}; Material(A)$)

a/h	w ([8])	w(present) (Geometric-nonlinear HSDT)
2	7.7661	7.7660
4	2.9091	2.90907
10	1.0900	1.0900
20	0.7760	0.77594
50	0.6838	0.68376
100	0.6705	0.67046

Center deflection for $(0^o/90^o)$ cross-ply S.S. square plate under UDL ($q_0 = 1.0; w = \frac{wE_2h^3 10^2}{q_0 a^4}; Material = (A)$)

Table 4.11: Deflection for simply-supp. boundary conditions of square plate material (A).

a/h	w([27])	w(present) (linear HSDT)
5	2.5813	2.5802
10	1.9192	1.9184
20	1.7479	1.7398
100	1.6789	1.6577

Table 4.12: First ply failure results for S.S. case.[36]

Stacking sequence	failure load for linear HSDT	failure load for geometric nonlinear HSDT	failure load for combined nonlinear HSDT	Max. deflection for combined nonlinear HSDT	First failed layer no.	Failed guass pt. no.
$(+45/-45/0/90)_{2s}$	1610	1569	1547	60	16	144
$(+45/-45/0/0)_{2s}$	1530	1478	1455	3.72	16	144
$(+45/-45)_{4s}$	1470	1440	1425	4.18	16	144
$(0/90)_{4s}$	2066	2040	2010	4.32	16	106

Chapter 5

Summary and Conclusions

5.1 Summary

The aim of present investigation has been to study the nonlinear response and the first ply failure of laminated plates under transverse loading. The overall investigation is based on the finite element formulation using the higher order shear deformation theory with a nine noded Lagrangian element having seven degrees of freedom per node. Green's strain vector imposed by Von Karman's assumptions is used to incorporate geometric nonlinearity.The governing finite element equations have been derived by using the total potential energy approach and then assembled to obtain a set of nonlinear algebraic equations.The equations have been solved by using Newton-Raphson method using incremental cum iterative procedure.The material non-linearity is also incorporated by using modified Ramberg-Osgood relations for normal stresses and for shear strains a cubical formula is used.The tangent modulii for various directions are obtained for each strain level and then stiffness matrix is

updated for each strain level. Picard's direct iterative procedure is used to solve the material non-linear case. For geometric and material nonlinear combined case, both these iterative techniques are used.Different types of laminates have been analysed for the simply-supported boundary conditions for maximum strain criterion only.The first failed layer number,the first failed guass point no. and the associated transverse deflections have also been found.

5.2 Conclusions

On the basis of the present study, the following important conclusions are made:

- The cross-ply laminates show more non-linearity than angle-ply laminates.
- By using combined non-linearity, cross-ply laminates behave to be stiffer than angle-ply laminates but also with increased non-linearity.
- The polynomial stress-strain relations can be used for the normal stress-strain case also instead of using the modified Ramberg-Osgood equations under the limits of strains not more than a particular value.Thus a good quantity of computational time can be saved.
- The cross-ply laminates has the largest strength among the various laminates considered.
- In general,the failure occurs near the corner of the laminate.
- The maximum transverse deflection under pure transverse load is greater for the linear HSDT than the geometric nonlinear HSDT.
- The maximum transverse deflection under pure transverse load is lesser for the combined non-linear case.

5.3 Scope for future work

Some problems of interest are:

- Buckling analysis with combined non-linearity.
- Dynamic analysis of laminates with combined loading.
- Dynamic stability analysis for combined non-linearity.
- Various analyses for different types of loadings.
- Various analyses under thermal loads with combined non-linearity.
- Progressive failure analysis of laminates under combined loads for combined non-linearity.
- Effect of presence of cut-outs on the strength and failure of laminates with combined non-linearity.

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